3 Special Types of Metric Spaces

3.1 Limits in the Real Numbers

The metric space given by the real numbers has a few properties that we can't generalize to all metric spaces, but that are still quite important.

The first important property of the real numbers is that they form a field—that is, we can do arithmetic with them. (Many of these results generalize to "normed vector spaces", which are vector spaces that are also metric spaces in a compatible way. Normed vector spaces are mostly outside the scope of this course, but would be important to a second course in analysis).

Proposition 3.1. *Let* $a_n, b_n \in \mathbb{R}$ *such that* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$. *Then*

1. $\lim_{n\to\infty}(a_n + b_n) = a + b$
2. $\lim_{n\to\infty}(a_n - b_n) = a - b$
3. $\lim_{n\to\infty}(a_n b_n) = ab$
4. *If* $\lim_{n\to\infty} b_n \neq 0$, *then* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.

Proof.

1. The goal is to show that if a_n is close to a, and b_n is close to b, then $a_n + b_n$ must be close to $a + b$.

   Let $\epsilon > 0$. Then there is a N_1 such that $|a_n - a| < \epsilon/2$ for all $n > N_1$, and there is a N_2 such that $|b_n - b| < \epsilon/2$ for all $n > N_2$.

   Let $N = \max\{N_1, N_2\}$. Then if $n < N$, we have

$$\begin{aligned} |a_n + b_n - (a+b)| &= |(a_n - a) + (b_n - b)| \\ &\leq |a_n - a| + |b_n - b| && \text{by the Triangle Inequality} \\ &< \epsilon/2 + \epsilon/2 = \epsilon && \text{because } n > N_1, N_2 \end{aligned}$$

2. Basically identical proof.
3. Exercise. (Hint: the quantity $a_n b$ may be helpful).
4. This one is a little trickier.

   First we prove that $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$. We see that

$$> \left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{|b - b_n|}{|b \cdot b_n|}.$$

The numerator can be made as small as we like since $\lim_{x\to\infty} b_n = b$; but we also need to make sure the *denominator* doesn't get too small.

But since b_n is close to b, we should be able to ensure that the denominator is bigger than $|b \cdot b/2| = b^2/2 > 0$. Then we just need to also make sure the numerator is smaller than $\epsilon b^2/2$.

Let $\epsilon > 0$. Then since $\epsilon b^2/2 > 0$, there is a N_1 such that $|b_n - b| < \epsilon b^2/2$ whenever $n > N_1$. And since $|b/2| > 0$, there is a N_2 such that $|b_n - b| < |b/2|$ whenever $n < N_2$.

Let $N = \max\{N_1, N_2\}$. Then if $n > N$, we have

$$
\begin{aligned}
|b_n - b| &< |b/2| && n > N_2 \\
|b/2| &> |b_n - b| \geq |b| - |b_n| && \text{Reverse Triangle Inequality} \\
|b_n| &> |b| - |b/2| = |b/2| \\
|bb_n| &> |b^2/2| = b^2/2 \\
\frac{1}{|bb_n|} &< \frac{1}{b^2/2} \\
\left|\frac{1}{b_n} - \frac{1}{b}\right| &= \frac{|b_n - b|}{|bb_n|} < \frac{|b_n - b|}{b^2/2} \\
&< \frac{\epsilon b^2/2}{b^2/2} = \epsilon && n > N_1.
\end{aligned}
$$

Thus $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$.

Now we can prove the original claim. We see that $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \frac{1}{b_n}$, and by the previous result this is equal to

$$\left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} \frac{1}{b_n}\right) = a\frac{1}{b} = \frac{a}{b}.$$

□

The other important property of the reals is that they are ordered and have the least upper bound property. This tells us a few things about sequence convergence.

Exercise 3.2. *If* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$*, and* $a_n \leq b_n$ *for all* n*, then* $a \leq b$*.*

Definition 3.3. We say a sequence (x_n) is *monotone increasing* or *monotonically increasing* if $x_n \leq x_{n+1}$ for every $n \in \mathbb{N}$. A sequence is *monotonically decreasing* if $x_n \geq x_{n+1}$ for every $n \in \mathbb{N}$.

Example 3.4. The sequence $(1/n)$ is monotonically decreasing.

The sequence $1, 1, 2, 2, 3, 3, \ldots$ is monotonically increasing.

The sequence $1, 1, 1, 1, \ldots$ is both monotonically increasing and monotonically decreasing.

Proposition 3.5 (Monotone Convergence). *Let (x_n) be a monotonically increasing sequence of real numbers that is bounded above. Then (x_n) converges.*

Proof. This is a straightforward application of the Least Upper Bound property. The set $\{x_n\}$ is bounded above and non-empty, so it has a least upper bound; set $x = \sup\{x_n\}$. We claim that $\lim_{n\to\infty} x_n = x$.

Let $\epsilon > 0$. Then since x is the least upper bound of $\{x_n\}$, there is some x_N such that $x_N > x - \epsilon$. Suppose $n > N$. Then by monotonicity, $x_n > x_N > x - \epsilon$, but since x is an upper bound, $x \geq x_n$. Thus

$$\begin{aligned} x - \epsilon < x_n &\leq x \\ -\epsilon < x_n - x &\leq 0 < \epsilon \\ |x_n - x| &< \epsilon. \end{aligned}$$

Thus by definition, $\lim_{n\to\infty} x_n = x$.

□

Exercise 3.6. *Let $S \subset \mathbb{R}$ be nonempty and bounded above. Prove there is a monotone sequence (x_n) such that $x_n \in S$ and $\lim_{n\to\infty} x_n = \sup S$.*

Remark 3.7. Proposition 3.5 tells us that a sequence must have a limit, but doesn't tell us what that limit is. (Well, it tells us that the limit is the supremum, but if we don't know the supremum already that doesn't really help). It's still extremely useful to show that various limits have to (or can't) exist.

This monotone convergence property is actually equivalent to the Least Upper Bound property: if any monotonically increasing sequence converges, then any set must have a least upper bound. This is the second version of "completeness" we'll see in this course. We'll discuss the third, and most general, next week.

Unfortunately, most sequences are not monotone sequences. Fortunately, we can kind of fake it.

Proposition 3.8. *If $a_n = \sup\{x_k : k \geq n\}$, then (a_n) is monotone decreasing.*

Similarly, if $b_n = \inf\{x_k : k \geq n\}$, then the sequence (b_n) is monotone increasing.

Proof. If $m > n$ then $\{x_k : k \geq m\} \subset \{x_k : k \geq n\}$, so $\inf\{x_k : k \geq m\} \geq \inf\{x_k : k \geq n\}$. Thus $a_m \geq a_n$. □

Example 3.9. The sequence $(1/n)$ is already monotone decreasing. We see that for every n, $a_n = \sup\{1/k : k \geq n\} = 1/n$. But $b_n = \inf\{1/k : k \geq n\} = 0$, so the sequence a_n is constantly 0.

If $(x_n) = 0, 1, 0, 1, \ldots$, then $b_n = \inf\{0, 1\} = 0$ and $a_n = \sup\{0, 1\} = 1$. These sequences are distinct and constant.

If $(x_n) = 1, 1, 1/2, 1, 1/3, 1, 1/4, 1, \ldots$, then

$$a_n = \sup\{1/k : k \geq n/2\} \cup \{1\} = 1$$
$$b_n = \inf\{1/k : k \geq n/2\} \cup \{1\} = 0.$$

Notice that not only are the sequences (a_n) and (b_n) different, they don't necessarily have the same limit. But their limits tell us about the limit of our original sequence (x_n).

Definition 3.10. Let (x_n) be a sequence of real numbers. We define the *limit inferior* of the sequence to be

$$\liminf_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n = \lim_{n\to\infty} b_n.$$

We define the *limit superior* to be

$$\limsup_{n\to\infty} x_n = \overline{\lim}_{n\to\infty} x_n = \lim_{n\to\infty} a_n.$$

The basic idea here is that the lim inf is the smallest value that gets hit infinitely often, and the lim sup is the largest value that gets hit infinitely often. These don't always exist—what if the sequence increases without bound?—but they almost always exist.

Proposition 3.11. *Let (x_n) be a bounded sequence. Then*

1. $\liminf x_n$ *and* $\limsup x_n$ *both exist.*
2. $\liminf_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} \left\{ \inf_{k\geq n} x_k \right\}$ *and* $\limsup_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} \left\{ \sup_{k\geq n} x_k \right\}$.
3. $\liminf x_n \leq \limsup x_n$.

Proof.
1. (a_n) is a bounded decreasing sequence, and (b_n) is a bounded increasing sequence. By the Monotone Convergence Theorem, both limits exist.

2. $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} b_n$ by definition. But (b_n) is monotone increasing, so this limit is $\sup(b_n)$. Since $b_n = \inf\{x_k : k \geq n\}$, the proposition follows.

3. For each n, we have $b_n \leq a_n$ since a_n is the supremum of a set and b_n is the infimum of that same set. (So, e.g., $b_n \leq x_n \leq a_n$). Thus by HW 5 problem 1, we have $\lim_{n\to\infty} b_n \leq \lim_{n\to\infty} a_n$.

□

Example 3.12. Let

$$x_n = \begin{cases} \frac{n+1}{n} & n \text{ odd} \\ 0 & n\text{even} \end{cases}$$

We have $b_n = \inf\{x_k : k \geq n\} = 0$, so $\liminf x_n = \lim_{n\to\infty} b_n = 0$.

We have $a_n = \sup\{x_k : k \geq n\} = \frac{n+1}{n}$ if n is odd, and $a_n = \frac{n+2}{n+1}$ if n is even. Then we claim $\limsup x_n = \lim_{n\to\infty} a_n = 1$. For $|x_n - 1| = \frac{1}{n}$ if n is odd, and $\frac{1}{n+1}$ if n is even.

If $\epsilon > 0$, we can take N so that $N > 1/\epsilon$ by the Archimedean property. Then if $n > N$, we have $|x_n - x| < \frac{1}{n} < \frac{1}{N} < \epsilon$.

The lim sup and lim inf are not actually limits of the sequence. But in many ways they behave the same. They follow the same limit laws that we proved for sequences in proposition 3.1. And they are *almost* limits of the sequence.

Proposition 3.13. *Let (x_n) be a bounded sequence. Then there exists a subsequence (x_{n_k}) such that $\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n$.*

Similarly, there exists a subsequence (x_{m_k}) such that $\lim_{k\to\infty} x_{m_k} = \liminf_{n\to\infty} x_n$.

Proof. We have $a_n = \sup\{x_k : k \geq n\}$ and set $x = \limsup x_n = \lim a_n$. We define x_{n_k} as follows:

Let $n_1 = 1$, that is, $x_{n_1} = x_1$. Define the rest of the sequence inductively: if we have defined x_{n_k}, we can choose some $m > n_k$ such that $|a_{n_k+1} - x_m| < \frac{1}{k+1}$, since $a_{n_k+1} = \sup\{x_m : m \geq n_{k+1}\}$. We define $n_{k+1} = m$ so that $x_{n_{k+1}} = n_m$.

We want to show that $\lim_{k\to\infty} x_{n_k} = x$. We first want to show that x_{n_k} gets close to a_{n_k}, and then we can use the fact that a_{n_k} gets close to x.

We know that $a_{n_k+1} \geq a_{n_k}$ since they're both supremums, and the first set contains the second. We also know that $a_{n_k} \geq x_{n_k}$, again since a_{n_k} is a supremum. Then we can calculate

$$\begin{aligned} |a_{n_k} - x_{n_k}| &= a_{n_k} - x_{n_k} \\ &\leq a_{n_{k-1}+1} - x_{n_k} < \frac{1}{k}. \end{aligned}$$

Now let $\epsilon > 0$. We know $\lim_{n\to\infty} a_n = x$, so $\lim_{k\to\infty} a_{n_k} = x$, and thus there is some N_1 such that if $k > N_1$ then $|a_{n_k} - x| < \epsilon/2$. There is also some N_2 so that $1/N_2 < \epsilon/2$, and thus if $k > N_2$ then $|a_{n_k} - x_{n_k}| < 1/k < 1/N_2 < \epsilon/2$.

Then if $k > \max\{N_1, N_2\}$ we have

$$\begin{aligned} |x - x_{n_k}| &= |x - a_{n_k} + a_{n_k} - x_{n_k}| \\ &\leq |x - a_{n_k}| + |a_{n_k} - x_{n_k}| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Thus we've constructed a subsequence (x_{n_k}) whose limit is $x = \limsup x_n$.

□

We want to show that the limit superior and limit inferior tell us about the limit of our sequence if it exists. But before we can do that, we need one more result that should be familiar from Calculus 1.

Exercise 3.14 (Squeeze Theorem)**.** *Let $(a_n), (b_n), (x_n)$ be sequences of real numbers such that $a_n \leq x_n \leq b_n$ for all $n \in \mathbb{N}$. Suppose (a_n) and (b_n) converge, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Then (x_n) converges, and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} a_n$.*

Remark 3.15. Notice that this is a bit stronger than just showing that if all three sequences converge, then the limit of x_n is between the limits of a_n and b_n. In particular, you have to show that if (x_n) is trapped between a_n and b_n then it has to approach a particular limit.

Proposition 3.16. *Let (x_n) be a bounded sequence of real numbers. Then (x_n) converges if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$. If (x_n) converges, then the limit is equal to the lim sup.*

Proof. Suppose $\lim_{n\to\infty} x_n = x$. Then by proposition 3.13 there is a subsequence (x_{n_k}) that converges to $\liminf_{n\to\infty} x_n$. But we know that $\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} x_n$ if the latter limit exists, so $\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$. Similarly, $\lim_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$, and thus the lim sup and lim inf are equal.

Conversely, suppose $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$. Then we have $b_n \leq x_n \leq a_n$ for each n, and we have

$$\begin{aligned} \lim_{n\to\infty} b_n &= \liminf_{n\to\infty} x_n \\ &= \limsup_{n\to\infty} x_n = \lim_{n\to\infty} a_n. \end{aligned}$$

Thus by the Squeeze Theorem 3.14, we know that (x_n) converges, and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} a_n = \limsup_{n\to\infty} x_n$$

□

The ability to make arguments using the least upper bound property and thee lim sup and lim inf in the real numbers is really important and useful. Over the next couple of sections we want to find ways to generalize these properties to other metric spaces.

3.2 Completeness

In this section we're working towards the idea of *completeness*, which generalizes the Least Upper Bound property. Recall the Least Upper Bound property tells us the real numbers don't have any "holes"—that everywhere we would hope to find a real number, there is one actually there.

To generalize this to metric spaces, we first have to ask where we *want* to find points. We can't use the same trick we used in the reals, because we don't have an order. Instead we do something like generalizing the Monotone Convergence Theorem: we want to see that every sequence that should converge does.

So what kind of sequences should converge?

Definition 3.17. Let (E, d) be a metric space, and let (x_n) be a sequence in E. We say that (x_n) is a *Cauchy sequence* if for every $\epsilon > 0$ there is a $N \in \mathbb{N}$ so that if $n, m > N$, then $d(x_n, x_m) < \epsilon$.

This definition looks similar to the definition of convergence, but is subtly and importantly different. A sequence converges if the values all get arbitrarily close to some limit point. A sequence is Cauchy if the values all get arbitrarily close to *each other*.

The basic idea here is that if the points of a sequence are getting arbitrarily close together, they should be gathering at one point—and we would like that point to be a limit.

Example 3.18. We claim the sequence $(1/n)$ is Cauchy under the absolute value metric. Let $\epsilon > 0$, and choose N so that $1/N < \epsilon/2$. Then if $m, n < N$, we have $|1/m - 1/n| \leq |1/m| + |1/n| < \frac{1}{N} + \frac{1}{N} < \epsilon$. Thus $(1/n)$ is Cauchy.

Notice what information we *didn't* use: we needed to know the metric, but we didn't need to know the space. We probably assumed that $(1/n)$ was a sequence in the real numbers, in which case it is convergent. If we think of it as a sequence of rational numbers, it also converges.

But if we think of it as a sequence in $(0, 1)$, it doesn't converge—because there's nothing there for it to converge to! But even though we can't say the sequence converges, we can still say it's Cauchy. Because the property of being Cauchy is internal to the sequence.

Example 3.19. We claim the sequence $0, 1, 0, 1, \ldots$ is not Cauchy. Let $\epsilon = 1$ and let $N \in \mathbb{N}$. Then there is some even $m > N$ so that $x_m = 1$, and there is some odd $n > N$ so that $x_n = 0$. Then $|x_m - x_n| = 1 \not< 1$.

Example 3.20. Consider the sequence $(x^n) \subset \mathcal{C}([0, 1], \mathbb{R})$ of continuous functions on the unit interval, under the L_1 metric. Is this Cauchy?

Let $\epsilon > 0$ and choose $N > 2/\epsilon$. Then if $m, n > N$, we have

$$\begin{aligned} d(x^m, x^n) &= \int_0^1 |x^m - x^n| \, dx \leq \int_0^1 |x^m| + |x^n| \, dx = \int_0^1 x^m + x^n \, dx \\ &= \frac{x^{m+1}}{m+1} + \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{m+1} + \frac{1}{n+1} < \frac{1}{N} + \frac{1}{N} < \epsilon. \end{aligned}$$

Thus this sequence is Cauchy.

Does this sequence converge? It does, in fact—just weirdly. We claim the limit is 0. Let $\epsilon > 0$ and let $N > 1/\epsilon$. Then if $n > N$, we have

$$\begin{aligned} d(x^n, 0) &= \int_0^1 |x^n - 0| \, dx = \int_0^1 x^n \, dx \\ &= \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1} \\ &< \frac{1}{N} < \epsilon. \end{aligned}$$

Thus $\lim_{n\to\infty} x^n = 0$.

(The weird part is: $1^n = 1$ for any n, but $0 = 0$, so the limit of the values at 1 is not the value of the limit at 1. We'll discuss this weirdness more later on in the course.)

Proposition 3.21. *Every convergent sequence is Cauchy.*

Proof. Suppose $\lim_{n\to\infty} x_n = x$. Let $\epsilon > 0$. Then there is some $N \in \mathbb{N}$ so that if $n > N$, then $d(x_n, x) < \epsilon/2$.

Now suppose $m, n > N$. Then $d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \epsilon/2 + \epsilon/2 = \epsilon$. Thus (x_n) is Cauchy. □

Exercise 3.22. *Prove that any Cauchy sequence is bounded.*

Exercise 3.23. *Let (x_n) be a Cauchy sequence. Then any subsequence of (x_n) is Cauchy.*

Proposition 3.24. *Let (x_n) be a Cauchy sequence, and suppose (x_{n_k}) is a convergent subsequence. Then (x_n) converges.*

Proof. Suppose $\lim_{k\to\infty} x_{n_k} = x$. Let $\epsilon > 0$. Then there is some $N_1 \in \mathbb{N}$ so that if $k > N_1$, then $d(x_{n_k}, x) < \epsilon/2$. And there is some $N_2 \in \mathbb{N}$ so that f $n, m > n$, then $d(x_n, x_m) < \epsilon/2$.

Let $N = \max\{N_1, N_2\}$, and let $k > N$. Then we know that $n_k \geq k > N$, so we know that $d(x_{n_k}, x) < \epsilon/2$. And we also have $k, n_k > N$, so $d(x_k, x_{n_k}) < \epsilon/2$. Thus $\epsilon > d(x_{n_k}, x) + d(x_k, x_{n_k}) \geq d(x, x_k)$ by the triangle inequality, and thus $\lim_{k\to\infty} x_k = x$. □

Definition 3.25. We say a metric space (E, d) is *complete* if every Cauchy sequence converges.

Example 3.26. $\mathbb{R}$ is complete under the absolute value metric.

Suppose (x_n) is a Cauchy sequence. Then (x_n) is bounded, and thus has an upper bound, and thus has a lim sup. Let $x = \limsup x_n$. We claim that $\lim_{n\to\infty} x_n = x$.

Let $\epsilon > 0$. By definition of Cauchy, there is a $N_1 \in \mathbb{N}$ so that $|x_m - x_n| < \epsilon/3$ whenever $m, n > N_1$.

Now consider the sequence $(a_n) = (\sup\{x_k : k \geq n\})$. We know that $\lim_{n\to\infty} a_n = \limsup x_n = x$, and thus there is a N_2 so that if $n > N_2$, then $|a_n - x| < \epsilon/3$.

Now let $N > N_1, N_2$, and suppose $n > N$. Then we know that $|a_n - x| < \epsilon/3$. Since $a_n = \sup\{x_k : k \geq n\}$, there is some $k \geq n$ such that $|a_n - x_k| < \epsilon/3$. And since $k > N$, we know that $|x_n - x_k| < \epsilon/3$. Adding these three inequalities together gives us

$$\epsilon > |x_n - x_k| + |x_k - a_n| + |a_n - x| \geq |x_n - x|$$

by the triangle inequality, and thus $\lim_{n\to\infty} x_n = x$.

Thus any Cauchy sequence converges, and so $\mathbb{R}$ is complete.

Remark 3.27. Rosenlicht has a slightly different proof that doesn't involve the work we did defining lim sup. It's substantially the same, though, just taking the upper bound of the set of lower bounds—which is, of course, the lim inf.

Example 3.28. The rational numbers $\mathbb{Q}$ are not complete. For instance, the sequence $3, 3.1, 3.14, 3.141, \ldots$ is Cauchy, since if $n, m > N$, they are the same to $N-1$ decimal places and thus $|x_n - x_m| < 10^{1-N}$. But it doesn't converge in $\mathbb{Q}$, since $\pi \notin \mathbb{Q}$.

Similarly, we saw in example 3.18 that the set $(0, 1)$ is not complete: the sequence $(1/n)$ is Cauchy, but it does not converge in $(0, 1)$.

Exercise 3.29. *If E is any metric space under the discrete metric, then it is complete.*

Proposition 3.30. *A closed subset of a complete metric space is complete.*

Proof. Let (E, d) be a complete metric space, and V a closed subset of E.

Suppose (x_n) is a Cauchy sequence in V. Then by definition of completeness, x_n converges in E, and thus there is an $x \in E$ so that $\lim_{n\to\infty} x_n = x$. But then x is the limit of a sequence in V, so $x \in V$ and thus x_n converges in V. □

Example 3.31. Any closed interval in $\mathbb{R}$ is complete.

In contrast, we saw that $(0, 1)$ was not complete in example 3.18. In fact, no non-trivial open interval will be complete.

There are many metric spaces that are complete; in general we prefer to only work in complete metric spaces. The complex numbers $\mathbb{C}$ are complete, as are the various spaces of funcitons L_1, L_2, L_∞ that we have discussed at various points (as long as you define them carefully).

But one more complete metric space will be a very important example in this course, so we write out the proof carefully:

Proposition 3.32. $\mathbb{R}^n$ *is complete for any* $n \in \mathbb{N}$.

We'll prove this under the sup metric, but the proof is similar for the sum or Euclidean metrics. In fact, it's not too hard to show that a sequence converges under the sup metric if and only if it converges under the Euclidean metric, if and only if it converges under the sum metric.

The hardest part of this proof is the notation. We'll prove the result in $\mathbb{R}^3$ to keep the notation simple and avoid having to write double subscripts on everything, but the proof is identical for the general case in $\mathbb{R}^n$.

Proof. Let $(p_n) = ((x_n, y_n, z_n))$ be a Cauchy sequence in $\mathbb{R}^3$. We claim that the three sequences of real numbers $(x_n), (y_n), (z_n)$ are all Cauchy, and then we will use the fact that $\mathbb{R}$ is complete to show that $\mathbb{R}^3$ is complete.

Let $\epsilon > 0$. Then there is a $N \in \mathbb{N}$ so that $d(p_n, p_m) < \epsilon$ if $n, m > N$. But $d(p_n, p_m) = \sup\{|x_n - x_m|, |y_n - y_m|, |z_n - z_m|\}$, and thus we have $|x_n - x_m|, |y_n - y_m|, |z_n - z_m| < \epsilon$. So by definition the sequences $(x_n), (y_n), (z_n)$ are all Cauchy sequences of real numbers.

Since $\mathbb{R}$ is complete, we know that there are $x, y, z \in \mathbb{R}$ so that $\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y, \lim_{n\to\infty} z_n = z$. Thus there is some $M_x, M_y, M_z \in \mathbb{N}$ so that if $n > M_x$ then $|x_n - x| < \epsilon$; we define M_y and M_z similarly.

Let $M = \max\{M_x, M_y, M_z\}$. Then if $n > M$ we have $|x_n - x| < \epsilon, |y_n - y| < \epsilon, |z_n - z| < \epsilon$, and thus

$$d(p_n, (x, y, z)) = \sup\{|x_n - x|, |y_n - y|, |z_n - z|\} < \epsilon$$

and thus $\lim_{n\to\infty} p_n = (x, y, z)$.

Thus any Cauchy sequence in $\mathbb{R}^3$ converges in $\mathbb{R}^3$, and so $\mathbb{R}^3$ is complete.

□

A final note on completeness: if (E, d) is a metric space, we define the *completion* of E to be the smallest metric space containing E that is complete. We can construct such a thing by taking the set of Cauchy sequences in E, and declaring two sequences equivalent if they go to "the same place"—which we can define by, say requiring that the sequence $x_1, y_1, x_2, y_2, \ldots$ still be Cauchy. Then we can define a metric on the set of equivalence classes of Cauchy sequences, and it is a complete metric space.

This is the third construction of the reals from the rationals: the reals are the completion of the rationals. So we can define the reals to be equivalence classes of Cauchy sequences of rationals. (In fact, this is almost what we did: we defined the reals to be the set of equivalence classes of infinite decimals, which is just a very specific type of Cauchy sequence).

3.3 Compactness

In this section we want to generalize the idea of lim sup and lim inf. Those operators are useful because they work on any bounded sequence: in $\mathbb{R}$, every bounded sequence "almost" has a limit. We want to see how far we can generalize that property.

Definition 3.33. Let (E, d) be a metric space, and $S \subset E$. We say that a point $x \in E$ is an *accumulation point* (or *cluster point*) of S if any open ball centered at x contains infinitely many points of S.

Example 3.34. 0 and 1 are accumulation points of $(0, 1)$. (1/2 is also an accumulation point of $(0, 1)$, in fact).

0 is an accumulation point of the set $\{1/n : n \in \mathbb{N}\}$.

Every real number is an accumulation point of $\mathbb{Q}$.

In $\mathbb{R}^2$ with the Euclidean metric, the set of accumulation points of $B_1(0, 0)$ is $\overline{B}_1(0, 0)$.

Proposition 3.35. *Let (E, d) be a metric space, and $S \subset E$. x is an accumulation point of S if and only if every open ball centered at x contains a point of $S \setminus \{x\}$.*

Proof. If x is an accumulation point of S, then every ball centered at x contains infinitely many points of S. Thus it contains at least two points, so it contains at least one point of $S \setminus \{x\}$.

Conversely, suppose every open ball centered at x contains a point of $S \setminus \{x\}$. Let $r > 0$, and suppose $B_r(x)$ contains only finitely many points (not including x) $y_1, \dots, y_n$. Then we can set $d = \min\{d(y_1, x), \dots, d(y_n, x)\}$. Then $B_d(x)$ contains no points of S (except possibly x), since otherwise we have a $y \in B_d(x) \subset B_r(x)$ with $d(y, x) < d$. But we know that every open ball contains some point in $S \setminus \{x\}$, so this is a contradiction. □

Proposition 3.36. *Let (E, d) be a metric space, and $S \subset E$. x is an accumulation point of S if and only if x is the limit of a sequence of points in $S \setminus \{x\}$.*

Proof. Suppose x is an accumulation point of S. We construct a sequence as follows: for each $n \in \mathbb{N}$, we know that $B_{1/n}(x)$ contains a point of $S \setminus \{x\}$, so take x_n to be such a point.

Then if $\epsilon > 0$ and $1/N < \epsilon$, if $n > N$ we know that $x_n \in B_{1/n}(x)$ so $d(x_n, x) < 1/n < 1/N < \epsilon$. Thus $\lim_{n\to\infty} x_n = x$. Thus x is the limit of a sequence in $S \setminus \{x\}$.

Conversely, suppose $x = \lim_{n\to\infty} x_n$ where $x_n \in S \setminus \{x\}$ for each n. Then for any $r > 0$, there is a $N \in \mathbb{N}$ so that $x_n \in B_r(x)$ for all $n > N$. Thus each open ball centered at x contains a point of $S \setminus \{x\}$, and thus x is an accumulation point for S. □

Exercise 3.37. *Let (E, d) be a metric space, and let $V \subset E$. Prove that V is closed if and only if it contains all its accumulation points.*

We're particularly interested in sequences here, so we want a similar definition for cluster points of sequences.

Definition 3.38. We say that x is an accumulation point of the sequence (x_n) every open ball centered at x contains x_n for infinitely many natural numbers n.

Example 3.39. 0 is an accumulation point for the sequence $(1/n)$.

The sequence $0, 1, 0, 1, 0, 1$ has two accumulation points: 0 and 1. (Notice that the set $\{0, 1, 0, 1, \dots\}$ is just the same as the set $\{0, 1\}$, and thus has no accumulation points at all as a set).

The sequence $0, 1, 2, 0, 1, 2$ has three accumulation points.

Proposition 3.40. *Let (x_n) be a sequence. x is an accumulation point of (x_n) if and only if it is the limit of some subsequence.*

Proof. Suppose x is the limit of a subsequence of x_n. Then there is some subsequence x_{n_k} so that $\lim_{k\to\infty} x_{n_k} = x$. Let $r > 0$; we need to show that $B_r(x)$ contains infinitely many x_n.

But by definition of convergence, there is some $N \in \mathbb{N}$ so that $d(x_{n_k}, x) < r$ for all $k > N$. Thus for every $k > N$, we know that$x_{n_k} \in B_r(x)$. There are infinitely many $k > N$, which proves the claim.

Conversely, suppose x is an accumulation point of (x_n). We need to construct a subsequence that approaches x. We do this just as we did in proposition 3.36; the details are left as an exercise.

□

Definition 3.41. We say a metric space (E, d) is *sequentially compact* if every sequence in E has an accumulation point in E.

Proposition 3.42. *A metric space is sequentially compact if and only if every infinite set has an accumulation point.*

Proof. Suppose (E, d) is a sequentially compact metric space, and $S \subset E$ is infinte. Let (x_n) be any sequence of distinct points in S. Then since (E, d) is sequentially compact, x_n has an accumulation point x. Let $r > 0$. Then there is some subsequence x_{n_k} and $N_r \in \mathbb{N}$ so that $x_{n_k} \in B_r(x)$ for all $k > N_r$, and thus B_r contains infinitely many points of S. Thus x is an accumulation point of S.

Conversely, suppose (E, d) is a metric space such that every infinite set has an accumulation point, and let (x_n) be a sequence. We need to show that (x_n) has an accumulation point.

We divide this into two cases. First, suppose the set $\{x_n\}$ is finite. Then there is at least one element x that appears infinitely many times in the sequence. We take the subsequence $x, x, x, \dots$, and this subsequence has x for a limit. Thus x is an accumulation point of the sequence.

Now, suppose the set $\{x_n\}$ is infinite. Then the set has an accumulation point x. We define a subsequence converging to x as follows: for each k, we can find infinitely many points x_m so that $x_m \in B_{1/k}(x)$. Thus in particular we can find one such that $m > n_{k-1}$; we set $n_k = m$.

Now we see that $\lim_{k\to\infty} x_{n_k} = x$. If $\epsilon > 0$, we set $1/N < \epsilon$, and then if $k > N$ we have $d(x_{n_k}, x) < 1/k < 1/N < \epsilon$. Thus (x_n) has a convergent subsequence, and thus an accumulation point. □

Corollary 3.43. *A metric space is sequentialy compact if and only if every sequence has a convergent subsequence.*

Proof. Suppose E is sequentially compact. Then every sequence has an accumulation point, which is the limit of some subsequence by proposition 4.7.

Conversely, suppose every sequence has a convergent subsequence. Then the limit of that convergent subsequence is an accumulation point of the sequence. □

Proposition 3.44. *Let (E, d) be a metric space, and let $S \subset E$ be a (non-empty) sequentially compact subset. Then S is closed and bounded.*

Proof. First we show S is closed. Let (x_n) be a convergent sequence in S that converges to some point $x \in E$. We know that (x_n) has an accumulation point in S, which we call y. We want to show that $x = y$.

Let $\epsilon > 0$. Then since $x = \lim_{n\to\infty} x_n$, there is some $N \in \mathbb{N}$ so that $d(x, x_n) < \epsilon/2$ for all $n > N$. But y is an accumulation point for (x_n), so there are infinitely many m such that $d(x_m, y) < \epsilon/2$. Then we can choose some m that is also greater than N; then we have $d(x_m, y) < \epsilon/2$ and $d(x_m, x) < \epsilon/2$.

By the triangle inequality, we see that $d(x, y) < \epsilon$. But this holds for any $\epsilon > 0$, so $d(x, y) = 0$ and thus $x = y$. So if (x_n) converges in E, its limit must be in S; thus S is closed.

Now we show that S is bounded. Let $x \in S$. If S is not bounded, then for each n there is a $x_n \in S$ such that $d(x, x_n) > n$. Then (x_n) is a sequence in S, and so it has an accumulation point y.

Let $r = d(x, y)$, and let $n > 2r$. Then $d(x, x_n) \leq d(x, y) + d(y, x_n)$ by the triangle inequality; but $2r < n < d(x, x_n)$ and $d(x, y) = r$, so we have $2r < r + d(y, x_n)$ and thus $r < d(y, x_n)$. But then $d(y, x_n) > r$ for all $n > 2r$, and thus $B_r(y)$ contains only finitely many x_n, which contradicts the claim that y is an accumulation point. Thus we have a contradiction, and thus S must be bounded. □

Proposition 3.45. *Let (E, d) be a sequentially compact metric space, and let $S \subset E$ be closed. Then S is sequentially compact.*

Proof. Let (x_n) be a sequence in S. Then (x_n) has an accumulation point in E; let x be an accumulation point of (x_n). Then x is the limit of some subsequence (x_{n_k}). But (x_{n_k}) is a sequence in S, and thus $\lim_{k\to\infty} x_{n_k} = x \in S$. Therefore (x_n) has an accumulation point in S. □

We've now seen some properties that any compact set must have—they all have to be closed and bounded. The converse of this theorem isn't quite true. But it's close!

Exercise 3.46. *If (x_n) is a bounded sequence of real numbers, then $\liminf_{n\to\infty} x_n$ is an accumulation point of (x_n), and $\limsup_{n\to\infty} x_n$ is an accumulation point of (x_n).*

Lemma 3.47. *Let V be a closed and bounded subset of $\mathbb{R}$. Then V is sequentially compact.*

Proof. Let (x_n) be a sequence in V. Then (x_n) is bounded, and so $\limsup_{n\to\infty} x_n$ is an accumulation point of x_n. Since V is closed, then $\limsup_{n\to\infty} x_n \in V$. So (x_n) has an accumulation point in V. □

Theorem 3.48 (Bolzano-Weierstrass)**.** *If V is a closed and bounded subset of $\mathbb{R}^n$, then V is sequentially compact.*

Proof. For notational reasons we'll only write down the proof for $\mathbb{R}^3$.

Let V be a closed and bounded subset of $\mathbb{R}^3$, and let (x_n, y_n, z_n) be a sequence in V. Then (x_n) is a bounded sequence in $\mathbb{R}$, so it has a convergent subsequence (x_{n_k}).

Now consider the sequence y_{n_k} which is a subsequence of (y_n). This is a bounded sequence in $\mathbb{R}$, and so it has some convergent subsequence y_{m_k}. Similarly, we now consider the sequence z_{m_k}, and it is a bounded sequence in $\mathbb{R}$ and so has a convergent subsequence z_{ℓ_k}.

Now consider the sequence $(x_{\ell_k}, y_{\ell_k}, z_{\ell_k})$ in V. We know that x_{ℓ_k} is a convergent sequence (since it is a subsequence of a subsequence of a convergent sequence), so there is some $x \in \mathbb{R}$ such that $\lim_{k\to\infty} x_{\ell_k} = x$, and some N_x so that if $k > N_x$ then $|x_{\ell_k} - x| < \epsilon$. Similarly, there is a $y \in \mathbb{R}$ and N_y so that if $k > N_y$ then $|y_{\ell_k} - y| < \epsilon$, and a $z \in \mathbb{R}$ and N_z so that if $k > N_z$ then $|z_{\ell_k} - z| < \epsilon$.

Let $N = \max\{N_x, N_y, N_z\}$. Then if $k > N$, we have that

$$d_{\sup}((x_{\ell_k}, y_{\ell_k}, z_{\ell_k}), (x, y, z) = \sup\{|x_{\ell_k} - x|, |y_{\ell_k} - y|, |z_{\ell_k}, z|\} < \epsilon.$$

Thus (x_n, y_n, z_n) has a convergent subsequence. And since V is closed, the limit must be an element of V. Thus every sequence in V has a convergent subsequence, and so by definition V is sequentially compact. □

We've shown that every closed and bounded subset of $\mathbb{R}^n$ is compact. But this isn't true in any metric space:

Example 3.49. Let S be any infinite subset of a discrete metric space. Then S is bounded since $S \subset B_2(x)$ for any $x \in S$. And S is closed since any subset of a discrete metric space is closed. But S is not compact, since we can take a sequence of distinct points of S and this sequence will have no accumulation points.

This example, like most discrete metric space examples, is a little weird. But you can even find counterexamples in a reasonable (normed vector space) metric space.

Example 3.50. We define the metric space $\ell_\infty(\mathbb{R})$ to be the set of bounded sequences of elements of R, and define the sup metric by

$$d_{\sup}((x_n),(y_n)) = \sup\{|x_n - y_n|\}.$$

Let $V = \overline{B}_1((0))$ be the closed ball of radius 1 centered at the sequence of all zeroes. This set is clearly closed and bounded.

Now consider the sequence defined by $e_n = (0,0,0,\dots,0,1,0,\dots)$ to be the sequence whose nth element is 1, and all of whose other elements are 0. This is a sequence in V. But for all distinct $m,n \in \mathbb{N}$, we see that $d_{\sup}(e_m,e_n) = 1$, so there is no x such that $d_{\sup}(e_m,x) < 1/2$ and also $d_{\sup}(e_n,x) < 1/2$. Thus the sequence (e_n) has no accumulation point; so V is not compact.

In general, it's surprisingly difficult to be compact in an infinite dimensional space. But it's totally possible; and a lot of work in functional analysis has to do with proving that some subset of a space of functions is compact. (This allows us to prove that differential equations have solutions, for instance).

There is one other way of thinking about compactness. It is easier to prove things with, but a bit harder to visualize.

Definition 3.51. Let (E,d) be a metric space and $S \subset E$. We say S is *(topologically) compact* if, whenever S is contained in union of open sets $S \subset \bigcup U$, then S is contained in the union of some finite collection of those open sets.

Example 3.52. A closed interval $[a,b] \subset \mathbb{R}$ is (topologically) compact.

Proposition 3.53. *Any topologically compact metric space is sequentially compact.*

Proof. Let (E,d) be topologically compact. By proposition 3.42, it's sufficient to prove that every infinite set has an accumulation point.

So suppose S is a subset of E with no accumulation point. Then for each $x \in E$, we can find some open ball $B_{r_x}(x)$ that contains only finitely many points of S.

Clearly, $E = \bigcup_{x\in E} B_{r_x}(x)$ since every point of E is in that union. By compactness, E is contained in some finite union of these balls; but each ball contains only finitely many points of S, so the finite union contains only finitely many points of S. But $S \subset E \subset \bigcup B_{r_x}(x)$, so S must be finite.

□

Remark 3.54. The converse of this theorem is also true: any sequentially compact space is topologically compact. This is a bit more irritating to prove, though. The basic idea is that if your space is not compact, then you can find a bad open cover, build a sequence that has only finitely many points in any of the open sets. But this sequence consequently has no accumulation point.