4 Continuous Functions

4.1 Limits of Functions

Definition 4.1. Let E, F be metric spaces, and let $f : E \to F$ be a function between them. Suppose a is a cluster point of E. Then we write $\lim_{x\to a} f(x) = b$, and say that b is the limit of f at a, if for every $\epsilon > 0$, there is a $\delta > 0$ such that if $0 < d(x, a) < \delta$ then $d(f(x), b) < \epsilon$.

Example 4.2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3x$. Then we claim $\lim_{x\to 2} f(x) = 6$.

Let $\epsilon > 0$ and let $\delta = \epsilon/3$. If $0 < d(x, 2) < \delta$, then $d(3x, 6) = |3x - 6| = 3|x - 2| < 3\delta = \epsilon$.

Example 4.3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Then we claim $\lim_{x\to 1} f(x) = 1$.

Let $\epsilon > 0$ and let $\delta \leq \underline{1, \epsilon/3}$. If $d(x, 1) < \delta$ then

$$\begin{aligned} d(x^2, 1) &= |x^2 - 1| = |x - 1| \cdot |x + 1| < \delta|x + 1| \\ &= \delta|x - 1 + 2| \leq \delta(|x - 1| + 2) < \delta(\delta + 2) \\ &\leq \delta(1 + 2) \leq 3(\epsilon/3) = \epsilon. \end{aligned}$$

Example 4.4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = \frac{x^2 y}{x^2+y^2}$. We claim that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.

Let $\epsilon > 0$. If $0 < d((x, y), (0, 0)) < \delta$, $x^2 + y^2 \neq 0$ so the function is defined. Then

$$\begin{aligned} d\left(\frac{x^2 y}{x^2 + y^2}, 0\right) &= \frac{|x^2 y|}{x^2 + y^2} = |y| \frac{x^2}{x^2 + y^2} \\ &\leq |y| \leq d((x, y), (0, 0)) < \delta. \end{aligned}$$

So if we take $\delta = \epsilon$, then $d((x, y), (0, 0)) < \delta$ implies that $|f(x, y) - 0| < \epsilon$.

The definition of a limit of a function is obviously similar to the definition of a limit of a sequence, but not identical. We'd like to avoid having to take the results we already proved about sequences, and prove them all over again for functions. Fortunately, we can prove one result relating sequences to functions, and then use it to bring over all our old sequence results.

Lemma 4.5. *Let $f : E \to F$, and let a be a cluster point of E. Then $\lim_{x\to a} f(x) = b$ if and only if, whenever (x_n) is a sequence that converges to a, then the sequence $f(x_n)$ converges to b.*

Proof. Suppose $\lim_{x\to a} f(x) = b$, and suppose (x_n) is a sequence in E such that $\lim_{n\to\infty} x_n = a$. Let $\epsilon > 0$. Then there is a $\delta > 0$ so that if $0 < d(x, a) < \delta$ then $d(f(x), b) < \epsilon$.

Since $\lim_{n\to\infty} x_n = a$, there is a $N \in \mathbb{N}$ so that if $n > N$ then $d(x_n, a) < \delta$, and thus $d(f(x_n), b) < \epsilon$. Thus $\lim_{n\to\infty} f(x_n) = b$.

Conversely, suppose that $\lim_{x\to a} f(x) \neq b$. Then this means there is some $\epsilon > 0$ such that, for any $\delta > 0$, there is some x such that $0 < d(x, a) < \delta$ but $d(f(x), b) \geq \epsilon$.

In particular, for each $n \in \mathbb{N}$ there is some x_n such that $0 < d(x_n, a) < 1/n$ but $d(f(x), b) \geq \epsilon$. By construction we see that $\lim_{n\to\infty} x_n = a$. But $d(f(x_n), b) \geq \epsilon$ for every n, so the sequence $(f(x_n))$ does not converge to b. This proves the claim.

□

Example 4.6. We claim $\lim_{x\to 0} \sin(1/x)$ does not exist.

Suppose that the limit does exist, and set $L = \lim_{x\to 0} \sin(1/x)$. First consider the sequence given by $x_n = \frac{1}{\pi/2+2n\pi}$. We see that $\lim_{n\to\infty} x_n = 0$, so $\lim_{n\to\infty} \sin(1/x_n) = L$. But $\sin(1/x_n) = \sin(\pi/2 + 2n\pi) = 1$, so $L = \lim_{n\to\infty} 1 = 1$.

Now consider the sequence $y_n = \frac{1}{3\pi/2+2n\pi}$. Then $\lim_{n\to\infty} y_n = 0$ and so

$$L = \lim_{n\to infty} \sin(1/y_n) = \lim_{n\to\infty} \sin(3\pi/2 + 2n\pi) = \lim_{n\to\infty} -1 = -1,$$

which is a contradiction.

Alternately, we could have argued as follows: Let $x_n = \frac{2}{n\pi}$. Then $\lim_{n\to\infty} x_n = 0$, so

$$L = \lim_{x\to 0} \sin(1/x) = \lim_{n\to\infty} \sin(1/x_n) = \lim_{n\to\infty} \sin(n\pi/2).$$

But $(\sin(n\pi/2)) = 1, 0, -1, 0, 1, \ldots$ has no limit, which is a contradiction.

Notice that this is essentially the same argument you would have seen in calculus 1 to prove that this limit does not exist. But we can make the argument much more simply by using the language of sequences.

We wish to pay some special attention to limits in the reals. (As with sequences, much of this work can be generalized to $\mathbb{R}^n$, but we won't do so here).

Proposition 4.7. *Let (E, d) be a metric space, and suppose f, g are functions from E to $\mathbb{R}$. If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$, then*

- $\lim_{x\to a} f(x) + g(x) = L_1 + L_2$
- $\lim_{x\to a} f(x) - g(x) = L_1 - L_2$
- $\lim_{x\to a} f(x)g(x) = L_1 L_2$
- *If $L_2 \neq 0$ then $\lim_{x\to a} f(x)/g(x) = L_1/L_2$.*

Proof. We could prove these directly, as we did with proposition 3.1 about limits of real sequences. But we can also just combine our result related sequence convergence to function limits with proposition 3.1 and avoid having to do any actual work.

Whenever (x_n) is a sequence converging to a, we know that $\lim_{n\to\infty} f(x_n) = \lim_{x\to a} f(x) = L_2$ and $\lim_{n\to\infty} g(x_n) = \lim_{x\to a} g(x) = L_2$. Thus

$$\lim_{n\to\infty} f(x_n) + g(x_n) = \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n) = L_1 + L_2$$

by proposition 3.1.

So we showed that whenever (x_n) converges to a, then $\lim_{n\to\infty} f(x_n) + g(x_n) = L_1 + L_2$. This proves that $\lim_{x\to a} f(x) + g(x) = L_1 + L_2$ by Lemma 4.5.

This proves the first statement; the others follow similarly.

□

4.2 Continuity

We are now ready to define the most important type of function we'll be talking about for the rest of the course.

Definition 4.8. Let E, F be metric spaces, and let $f : E \to F$. Let $a \in E$ be an accumulation point of E. We say that f is *continuous at a* if $\lim_{x\to a} f(x) = f(a)$.

We say f is *continuous* (or continuous on its domain, or continuous on E) if f is continuous at every accumulation point of E.

Example 4.9. Any rational function is continuous on its domain, by Proposition 4.7.

Example 4.10. Let $f(x) = \begin{cases} x^3 & x < 0 \\ x^2 & x \geq 0 \end{cases}$. We claim $f(x)$ is continuous.

For $a < 0$, we know that $\lim_{x\to a} f(x) = \lim_{x\to a} x^3 = a^3 = f(a)$; for $a > 0$ we know that $\lim_{x\to a} f(x) = \lim_{x\to a} x^2 = a^2 = f(a)$.

So we just need to show that $\lim_{x\to 0} f(x) = f(0) = 0$. Let $\epsilon > 0$. We can do this explicitly or more abstractly.

Explicit argument: If $\delta < \sqrt{\epsilon}$ and $d(x, 0) < \delta$ then we have $d(x^2, 0) = x^2 < \delta^2 < \epsilon$. And if $\delta < \sqrt[3]{\epsilon}$ then $d(x^2, 0) = |x^3| < \delta^3 < \epsilon$. So set $\delta < \sqrt{\epsilon}, \sqrt[3]{\epsilon}$. Then if $d(x, 0) < \delta$, then $d(f(x), 0) < \epsilon$ since either $f(x) = x^2$ or $f(x) = x^3$. This proves that $\lim_{x\to 0} f(x) = 0$.

Abstract argument: We know that $\lim_{x\to 0} x^3 = 0^3 = 0$, so there is a δ_1 so that if $d(x, 0) < \delta_1$ then $d(x^3, 0) < \epsilon$. And we know that $\lim_{x\to 0} x^2 = 0^2 = 0$, so there is a δ_2 so that if $d(x, 0) < \delta_2$ then $d(x^2, 0) < \epsilon$.

Let $\delta = \min\{\delta_1, \delta_2\}$. If $d(x, 0) < \delta$ then $d(x^2, 0) < \epsilon$ and $d(x^3, 0) < \epsilon$, and thus $d(f(x), 0) < \epsilon$. This proves that $\lim_{x\to 0} f(x) = 0$.

Example 4.11. Define a function $\chi_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ by $\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. We claim that f is discontinuous at every real number.

First, suppose $a \notin \mathbb{Q}$. Let $\epsilon = 1$. For any $\delta > 0$, we can find a rational number x such that $d(x, a) < \delta$. In particular, there is a $N \in \mathbb{N}$ such that $1/N < \delta$, and there is a $n \in \mathbb{N}$ so that $n/N < a < (n + 1)/N$. Then $d(n/N, a) < 1/N < \delta$.

Then for any $\delta > 0$, there is a x with $d(x, a) < \delta$ but $d(f(x), f(a)) = d(1, 0) = 1 \not< \epsilon$. Thus $\lim_{x\to a} f(x) \neq f(a)$ and thus f is not continuous at a.

Now suppose $a \in \mathbb{Q}$. Let $\epsilon = 1$. For every $\delta > 0$, we claim there is an irrational number x such that $d(x, a) < \delta$: Let b be any positive rational number. Then if $N > b/\delta$ we have $b/N < \delta$ and $a + b/N$ is irrational. Let $x = a + b/N$. Then $d(x, a) < \delta$, but $f(x) = 0$ and $f(a) = 1$ so $d(f(x), f(a)) = 1 \not< \epsilon$. Thus $\lim_{x\to a} f(x) \neq f(a)$ and f is not continuous at a.

Exercise 4.12. *If $f : E \to F$ is a continuous function and (x_n) is a convergent sequence in E such that $\lim_{n\to\infty} x_n = x$, then prove that $\lim_{n\to\infty} f(x_n) = f(x)$.*

We can rephrase the definition of continuity in terms of open balls or open sets. Recall the definition of inverse image:

Definition 4.13. Let $f : E \to F$ be a function, and let $U \subset F$. We define $f^{-1}(U) = \{x \in E : f(x) \in U\}$.

This definition makes sense even if f is not invertible.

Example 4.14. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $x \mapsto x^2$. Then

$$
\begin{aligned}
f^{-1}(\{0\}) &= \{0\} & f^{-1}(\{1\}) &= \{1, -1\} \\
f^{-1}(\{-1\}) &= \varnothing & f^{-1}(-2, -1) &= \varnothing \\
f^{-1}([0, 1]) &= [-1, 1] & f^{-1}((0, +\infty)) &= \mathbb{R} \setminus \{0\}.
\end{aligned}
$$

Proposition 4.15. *Let $f : E \to F$ be a function of metric spaces. f is continuous (at every point in E) if and only if, whenever $U \subset F$ is open, then $f^{-1}(U) \subset E$ is also open.*

Proof. Suppose f is continuous, and suppose $U \subset F$ is open. We need to show that $f^{-1}(U)$ is open. So let $a \in f^{-1}(U)$. We want to find an open ball containing a that is contained in $f^{-1}(U)$.

Then $f(a) \in U$, and U is open, so there is a ϵ such that $B_\epsilon(f(a)) \subset U$. But f is continuous, so $\lim_{x\to a} f(x) = f(a)$. Thus there is a $\delta > 0$ so that whenever $d(x, a) < \delta$, then $d(f(x), f(a)) < \epsilon$.

So we claim that $B_\delta(a) \subset f^{-1}(U)$. If $x \in B_\delta(a)$, then $d(x, a) < \delta$, so $d(f(x), f(a)) < \epsilon$, and thus $f(x) \in B_\epsilon(f(a)) \subset U$. But since $f(x) \in U$ we see that $x \in f^{-1}(U)$ by definition. Thus $B_\delta(a) \subset f^{-1}(U)$. We can find such a $B_\delta(a)$ for any $a \in f^{-1}(U)$, so we see that $f^{-1}(U)$ is open.

Conversely, suppose we know that $f^{-1}(U)$ is open for any open U. Let $a \in E$; we want to show that $\lim_{x\to a} f(x) = f(a)$.

Let $\epsilon > 0$. Then the ball $B_\epsilon(f(a))$ is open in F, so $f^{-1}(B_\epsilon(f(a)))$ is open in E. Clearly $a \in f^{-1}(B_\epsilon(f(a)))$ since $f(a) \in B_\epsilon(f(a))$, and thus there is a $\delta > 0$ such that $B_\delta(a) \subset f^{-1}(B_\epsilon(f(a)))$.

Then if $d(x, a) < \delta$, we have $x \in B_\delta(a) \subset f^{-1}(B_\epsilon(f(a)))$ and thus $f(x) \in B_\epsilon(f(a))$. So $d(f(x), f(a)) < \epsilon$. Thus $\lim_{x\to a} f(x) = f(a)$, and f is continuous at a. □

Corollary 4.16. *If $f : E \to \mathbb{R}$ is continuous and $a \in \mathbb{R}$, then the sets*

$$\{x : f(x) > a\} \qquad \{x : f(x) < a\}$$

are open.

Exercise 4.17. *If $f : E \to F$ and $g : F \to G$ are continuous functions of metric spaces, prove that $g \circ f$ is continuous. (Hint: use the topological result about open sets, not the limit definition).*

We're particularly interested in functions involving the real numbers. We can make two easy observations here.

Exercise 4.18. *Let $p_i : \mathbb{R}^n \to \mathbb{R}$ be the projection into the ith coordinate given by $p_i(x_1, \dots, x_n) = x_i$. Prove that p_i is a continuous function.*

Proposition 4.19. *Let E be any metric space, and $f : E \to \mathbb{R}^n$. Then f is continuous at $a \in E$ if and only if $p_i \circ f$ is continuous at a for each i.*

Proof. If f is continuous, then $p_i \circ f$ is a composition of continuous functions, and thus continuous.

Conversely, suppose $p_i \circ f$ is a continuous function. Let $\epsilon > 0$. Then for each i, there is a δ_i so that if $d(x, a) < \delta_i$, then $|p_i(f(x)), p_i(f(a))| < \epsilon$.

Let $\delta = \min\{\delta_i\}$ (which is well-defined since n is finite). Then if $d(x,a) < \delta$, for each i we have $d(x,a) < \delta_i$ and thus $|p_i(f(x)), p_i(f(a))| < \epsilon$. Then

$$d(f(x), f(a)) = \sup\{|p_i(f(x)) - p_i(f(a))|\} < \epsilon.$$

Thus $\lim_{x\to a} f(x) = f(a)$. This is true for any $a \in E$, so f is continuous at a. □

Remark 4.20. This proof is pretty trivial in the sup metric. It's a bit trickier in the Euclidean or sum metrics, but essentially the same—you just have to use $\epsilon/\sqrt{n}$ or ϵ/n.

Example 4.21. The function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x,y,z) = x^3y + xz^2 - yz$ is continuous.

The function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = \frac{x^2}{x^2+y^2}$ is continuous everywhere except the origin (where it is not defined).

4.3 Compact Sets and the Extreme Value Theorem

We said that continuous functions behave well with respect to metric spaces. They also behave well with regard to specifically compact spaces.

Proposition 4.22. *Let $f : E \to F$ be a continuous function on metric spaces. If E is compact, then the image $f(E) = \{f(x) : x \in E\}$ is also compact.*

Proof. We'll prove this for sequential compactness.

Suppose $(f(x_n))$ is a sequence in $f(E)$. (Every sequence looks like this, since $f(E) = \{f(x) : x \in E\}$. Then x_n is a sequence in E, and since E is compact, it has a convergent subsequence (x_{n_k}), and there is some x such that $\lim_{k\to\infty} x_{n_k} = x$.

But since f is continuous, we know that $\lim_{n\to\infty} f(x_{n_k}) = f(x)$, and thus $(f(x_n))$ has a convergent subsequence. Since every sequence in $f(E)$ has a convergent subsequence, we know that $f(E)$ is sequentially compact. □

Alternate topological proof. We can also prove this result using the topological definition. I like this proof better, but it's a bit more abstract. (It's not actually any more complicated; but it looks more complicated because we've spent much less time working with these concepts).

Suppose $f(E) \subset \bigcup U_\alpha$. Then for each $x \in E$, we have $f(x) \in U_\alpha$ for some α, and thus $x \in \bigcup f^{-1}(U_\alpha)$. So we have $E \subset \bigcup f^{-1}(U_\alpha)$.

But since f is continuous, we know that $f^{-1}(U_\alpha)$ is open for each α. So we've written E as a subset of a union of open sets; thus there is some finite collection $f^{-1}(U_{\alpha_1}), \dots, f^{-1}(U_{\alpha_n})$ so that $E \subset f^{-1}(U_{\alpha_1}) \cup \cdots \cup f^{-1}(U_{\alpha_n})$.

But then we have that for each $x \in E$, $f(x) \in U_{\alpha_i}$ for some i. Thus $f(E) \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$. So whenever $f(E)$ is contained in a union of open sets, it is contained in the union of some finite collection of them, and thus $f(E)$ is compact.

□

This seems like a weird technical result, and to some extent it is. But it's a technical result that has a lot of easy and powerful implications.

Definition 4.23. Let $f : E \to F$ be a function where F is a metric space. We say f is *bounded* if the set $f(E)$ is bounded, that is, contained in some ball.

Corollary 4.24. *Let $f : E \to F$ be a continuous function of metric spaces. If E is compact, then f is a bounded function.*

We often want to require our functions to be bounded. This allows us to talk about their suprema, as we do in the metric space L_∞ which we will discuss soon. It also is a necessary ingredient to many proofs that integrals are well-defined and other similar results.

Corollary 4.25 (Extreme Value Theorem)**.** *Let $f : E \to \mathbb{R}$ be a continuous function, and E a non-empty compact metric space. Then f attains a maximum value at some point, and a minimum value at some point.*

Proof. $f(E)$ is a compact set, and thus closed and bounded. Since it is bounded, it has an infimum and a supremum; since it is closed, the infimum and supremum must be elements of $f(E)$. Thus there exist x_1, x_2 such that $f(x_1) = \sup f(E)$ and $f(x_2) = \inf f(E)$, and thus f attains a maximum and a minimum value. □

Remark 4.26. The compactness condition is really important here. For instance, $f(x) = x$ is continuous, but doesn't achieve a maximum on the set $(0, 1)$, which is bounded.

4.4 Connected sets and the Intermediate Value Theorem

In high school you may have heard the following "definition" of continuous functions: f is continuous if you can draw its graph without picking up your pen. This is a fairly inadequate definition for our purposes, both because it doesn't generalize well to higher dimensions and other metric spaces, and because it isn't terribly precise. However, we'd still like to prove that it is *true*.

To do this we need to introduce the idea of connectedness. Intuitively, we say a set is connected if it isn't made up of two independent pieces; we can't separate it without cutting or breaking or tearing something. WE can formalize that idea like this:

Definition 4.27. Let (E, d) be a metric space. We say that E is *connected* if it is not the disjoint union of two non-empty open sets.

Example 4.28. The metric space $(0, 1) \cup (2, 3)$ is not connected.

The metric space $[0, 1] \cup [2, 3]$ is not connected, because $[0, 1]$ and $[2, 3]$ are open in that metric space.

No discrete metric space with more than one point is connected, since any subset of the discrete metric space is open.

Lemma 4.29. *Any subset $S \subset \mathbb{R}$ that contains two distinct points a, b, and does not contain all of $[a, b]$, is not connected.*

Proof. Suppose $c \in (a, b)$ and $c \notin S$. Then we can take $T_1 = \{x \in S : x < c\}$ and $T_2 = \{x \in S, x > c\}$. Then T_1 and T_2 are disjoint open subsets of S, and $S = T_1 \cup T_2$. □

It's pretty difficult to prove a metric space is connected from this definition, but there's a slightly different way to look at it that is much easier to deal with.

Proposition 4.30. *A metric space (E, d) is disconnected if and only if it contains a non-trivial set (other than $\varnothing$ or E) that is both closed and open.*

Proof. If S is both closed and open, then S^C is also open, and thus $E = S \cup S^C$ is a disjoint union of open sets. Since $S \neq \varnothing$ it is nonempty, and since $S \neq E$ we know S^C is non-empty.

Conversely, suppose $E = S \cup T$ is a disjoint union of non-empty open sets. Then S is open, and since T is open S is also closed. Further, $S \neq \varnothing$ because it is non-empty, and $S \neq E$ since T is non-empty. □

Example 4.31. $\mathbb{R}$ is connected because no subset can be closed and open simultaneously. Similarly, $\mathbb{R}^n$ is connected.

Lemma 4.32. *Any closed interval in $\mathbb{R}$ is connected.*

Proof. Let $[a, b] \subset \mathbb{R}$, and suppose we have a non-empty set $S \subsetneq [a, b]$ that is both closed and open. We can assume without loss of generality that $b \notin S$, since if $b \in S$ we can consider the non-trivial closed and open set S^C instead.

Then S is closed and bounded, and thus has a maximum element $c < b$. But S is open, so it must contain some open ball centered at c, which will contain elements of $[a, b]$ which are larger than c, contradicting the maximality of c. □

Lemma 4.33. *Let $S \subset \mathbb{R}$ have the property that if $a, b \in S$, then $[a, b] \subset S$. Then S is connected.*

Proof. Suppose S is the disjoint union of two non-empty open sets A, B. Then there exist $a \in A, b \in B$, and we can assume without loss of generality that $a < b$. Then $A \cap [a, b]$ and $B \cap [a, b]$ are open in $[a, b]$, and thus $[a, b]$ is disconnected, contradicting the previous lemma. □

Corollary 4.34. $\mathbb{R}$ *is connected.*

Any open interval in $\mathbb{R}$ *is connected.*

Remark 4.35. We can use this to prove that if a metric space is *path-connected*—that is, any two points are connected by some parametrized curve—then the space is also connected. If we can write the space as a union of two disjoint open sets, then we can write the path as the union of two disjoint open sets, which then allows us to write the pre-image of the path as a union of two disjoint open sets. But the pre-image is a closed interval, so this is a contradiction.

This proves, among other things, that $\mathbb{R}^n$ is connected. It's also a bit outside the scope of this course.

Now let's apply our theory of continuous functions to connected sets. We know intuitively that continuous functions shouldn't break things apart, and thus they shouldn't turn connected sets into disconnected sets. Fortunately, this turns out to be correct.

Lemma 4.36. *Let $f : E \to F$ be a continuous function. If E is connected, then so is $f(E)$.*

Proof. Suppose we can write $f(E) = S \cup T$ where S, T are disjoint non-empty open sets. Then for every $x \in E$ we have either $f(x) \in S$ or $f(x) \in T$, but not both. So $E \subset f^{-1}(S) \cup f^{-1}(T)$ is a disjoint union.

Further, if $y \in S$ then there is some $x \in E$ such that $f(x) = y$, so $f^{-1}(S)$ is non-empty, and a similar argument shows that $f^{-1}(T)$ is non-empty. But f is continuous, so $f^{-1}(S)$ and $f^{-1}(T)$ are both open; and thus E is the disjoint union of non-empty open sets, and thus is disconnected, which is a contradiction. □

Corollary 4.37 (Intermediate Value Theorem)**.** *Let (E, d) be a metric space, and $f : E \to \mathbb{R}$ a continuous function. If $a, b \in E$ with $f(a) < f(b)$, and $y \in (f(a), f(b))$, then there is a $c \in E$ such that $f(c) = y$.*

Proof. Since f is continuous and E is connected, we know $f(E)$ is connected. Since $f(a), f(b) \in f(E)$, by lemma 4.29 we know that $[f(a), f(b)] \subset f(E)$. But then $y \in f(E)$ so there is some $c \in E$ such that $f(c) = y$. □

Corollary 4.38 (Intermediate Value Theorem for Real Functions)**.** *Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function. If* y *is between* $f(a)$ *and* $f(b)$ *then there is a* $c \in (a, b)$ *such that* $f(c) = y$*.*

We can get one more corollary out of this, which becomes very important in numerical analysis and other computational branches of math.

Definition 4.39. Let $f : E \to E$ be a function. We say that x is a *fixed point* of f if $f(x) = x$.

Exercise 4.40. *Suppose* $f : [0, 1] \to [0, 1]$ *is continuous. Show that* f *has a fixed point. That is, show there is an* $x \in [0, 1]$ *such that* $f(x) = x$*.*