5 Integrals of Real Functions

5.1 Riemann Sums

Definition 5.1. Let $a, b \in \mathbb{R}$ with $a < b$. We define a *partition* of the closed interval $[a, b]$ to be a finite set of numbers $\{x_0, \ldots, x_n\}$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. The *width* of a partition is

$$\max\{x_i - x_{i-1} : 1 \le i \le n\}.$$

Definition 5.2. If $f : [a, b] \to \mathbb{R}$, and $P = \{x_0, \ldots, x_n\}$ is a partition of $[a, b]$, then we define a *Riemann sum for f corresponding to P* to be

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$

where $x_{i-1} \le x_i^* \le x_i$ for each $1 \le i \le n$.

Definition 5.3. Let $f : [a, b] \to \mathbb{R}$. We say f is *Riemann Integrable on* $[a, b]$ if there is a number $I \in \mathbb{R}$ so that, for any $\epsilon > 0$, there is a $\delta > 0$ such that, if S is a Riemann sum corresponding to a partition of width less than δ, then $|S - I| < \epsilon$. In this case we say that I is the *Riemann Integral of f* and write $I = \int_a^b f(x)\,dx$.

Remark 5.4. If f is defined on some larger set containing $[a, b]$, this definition still works; we define $\int_a^b f(x)\,dx$ to be the integral of the restriction of f to the domain $[a, b]$.

The Riemann integral is not a limit of a sequence or a function; it is an entirely different type of limit. (Technically, all of these limits are generalized by the concept of a *topological net*, but we're not really going to go there). The big difference here is that there are lots of different partitions of any given width, and they're not in any particular "order" with respect to each other. For our purposes we can only ask that the width be small enough, and we have to ask that this is enough.

Exercise 5.5. *Prove that the Riemann integral is unique. That is, if $f : [a, b] \to \mathbb{R}$ and both I and J satisfy the definition of Riemann integral of f, then prove that $I = J$.*

Example 5.6. Let $f : [a, b] \to \mathbb{R}$ be defined by $f(x) = c$ for some fixed $c \in \mathbb{R}$. We claim that $\int_a^b f(x)\,dx = c(a - b)$.

Let $\epsilon > 0$. Then any Riemann sum for f is

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) = \sum_{i=1}^{n} c(x_i - x_{i-1}) = c\sum_{i=1}^{n}(x_i - x_{i-1}) = c(x_n - x_0) = c(b - a).$$

Thus we can take δ to be any positive real number, and for any partition of width less than δ we have $|S - c(b-a)| = 0 < \epsilon$ and thus $c(b-a) = \int_a^b f(x)\,dx$ by definition.

Exercise 5.7. *Let $c \in (a,b)$, and let $f : [a,b] \to \mathbb{R}$ be defined by $f(c) = 1$ and $f(x) = 0$ if $x \neq c$. Prove that $\int_a^b f(x)\,dx = 0$.*

Example 5.8. Let $c, d \in [a,b]$ with $a < c < d < b$. Define $f(x) = \chi_{(c,d)}(x) = \begin{cases} 1 & x \in (c,d) \\ 0 & x \notin (c,d) \end{cases}$. We claim that $\int_a^b f(x)\,dx = d - c$.

Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of width less than δ. There exist r, s such that $x_r \leq c \leq x_{r+1}$ and $x_s \leq d \leq x_{s+1}$. Then for any x_i^* with $i \leq r$ or $i > s+1$, we have $f(x_i^*) = 0$, and for any x_i^* with $r + 1 < i \leq s$ we have $f(x_i^*) = 1$.

So if S is a Riemann sum corresponding to P, we have

$$\begin{aligned}
S &= \sum_{i=1}^n f(x_i^*)(x_i - x_{i-1}) \\
&= \sum_{i=1}^r f(x_i^*)(x_i - x_{i-1}) + f(x_{r+1}^*)(x_{r+1} - x_r) + \sum_{i=r+2}^s f(x_i^*)(x_i - x_{i-1}) \\
&\qquad + f(x_{s+1}^*)(x_{s+1} - x_s) + \sum_{i=s+2}^n f(x_i^*)(x_i - x_{i-1}) \\
&= \sum_{i=1}^r 0(x_i - x_{i-1}) + f(x_{r+1}^*)(x_{r+1} - x_r) + \sum_{i=r+2}^s 1(x_i - x_{i-1}) \\
&\qquad + f(x_{s+1}^*)(x_{s+1} - x_s) + \sum_{i=s+2}^n 0(x_i - x_{i-1}) \\
&= f(x_{r+1}^*(x_{r+1} - x_r) + f(x_{s+1}^*)(x_{s+1} - x_s) + (x_s - x_{r+1}).
\end{aligned}$$

Thus in particular, we see that $x_s - x_{r+1} \leq S \leq x_{s+1} - x_r$ and so, subtracting $d - c$ through the chain of inequalities, we get

$$x_s - d - (x_{r+1} - c) \leq S - (d-c) \leq x_{s+1} - d - (x_r - c).$$

Since the partition has width less than δ, we know that $d - x_s, x_{r+1} - c, x_{s+1} - d, x_r - c < \delta$, and thus we have $-2\delta < S - (d-c) < 2\delta$, and so $|S - (d-c)| < 2\delta$.

So if $\epsilon > 0$, we can take $\delta < \epsilon/2$, and then if P is a partition of width less than δ, we have $|S - (d-c)| < 2\delta < \epsilon$. Thus by definition $\int_a^b f(x)\,dx = d - c$.

Example 5.9. Let $\chi_{\mathbb{Q}} : [a,b] \to \mathbb{R}$ be the characteristic function of the rationals, defined by $\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ We claim that $\int_a^b \chi_{\mathbb{Q}}(x)\,dx$ does not exist.

Let $P = \{x_0, \dots, x_n\}$ be any partition of $[a, b]$. Each interval $[x_{i-1}, x_i]$ contains a rational number, so we can take each x_i^* to be rational. Then the corresponding Riemann sum is

$$S_1 = \sum_{i=1}^{n} \chi_{\mathbb{Q}}(x_i^*)(x_i - x_{i-1}) = \sum_{i=1}^{n} 1(x_i - x_{i-1}) = x_n - x_0 = b - a.$$

But it is also true that each interval $[x_{i-1}, x_i]$ contains an irrational number, so we can take each x_i^* to be irrational. Then the corresponding Riemann sum is

$$S_2 = \sum_{i=1}^{n} \chi_{\mathbb{Q}}(x_i^*)(x_i - x_{i-1}) = \sum_{i=1}^{n} 0 \cdot (x_i - x_{i-1}) = 0.$$

Then it is clear that no Riemann integral exists. Suppose $I = \int_a^b \chi_{\mathbb{Q}}(x)\,dx$ exists, and let $\epsilon = (b-a)/2$. Then for any $\delta > 0$, we can find a partition of width less than δ, and by the above argument we have $|(b-a) - I| < \epsilon$ and $|0 - I| < \epsilon$, which by the triangle inequality gives us $|b - a| < 2\epsilon = (b-a)/2$ which is a contradiction.

5.2 Integral Properties and Linearity

In this section, we want to establish and prove a few important properties of the integral. These properties will hold for any integrals that do in fact exist; but we don't yet have an easy way of showing that integrals do exist. That will come in the next two sections.

Proposition 5.10 (Linearity). *The Riemann Integral is a linear function from the vector space of integrable functions on $[a, b]$ to $\mathbb{R}$. That is,*

1. *If $f, g : [a, b] \to \mathbb{R}$ are integrable functions, then $f + g$ is integrable, and*

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

2. *If $f : [a, b] \to \mathbb{R}$ is an integrable function and $c \in \mathbb{R}$, then $cf(x)$ is integrable and*

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

Proof. 1. Let $\epsilon > 0$. Then there are δ_1, δ_2, so that if a partition $P = \{x_0, \dots, x_n\}$ has width less than δ_1 then $\left|\sum_{i=1}^n f(x_i^*)(x_i - x_{i-1}) - \int_a^b f(x)\,dx\right| < \epsilon/2$, and similarly if P has width less than δ_2 then $\left|\sum_{i=1}^n g(x_i^*)(x_i - x_{i-1}) - \int_a^b g(x)\,dx\right| < \epsilon/2$.

Let $\delta = \min \delta_1, \delta_2$. Then if P has width less than δ, we see that

$$\begin{aligned}
&\left| \sum_{i=1}^n \left(f(x_i^*) + g(x_i^*)\right)(x_i - x_{i-1}) - \left(\int_a^b f(x)\,dx + \int_a^b g(x)\,dx \right) \right| \\
&\qquad = \left| \sum_{i=1}^n f(x_i^*)(x_i - x_{i-1}) - \int_a^b f(x)\,dx + \sum_{i=1}^n g(x_i^*)(x_i - x_{i-1}) - \int_a^b g(x)\,dx \right| \\
&\qquad \leq \left| \sum_{i=1}^n f(x_i^*)(x_i - x_{i-1}) - \int_a^b f(x)\,dx \right| + \left| \sum_{i=1}^n g(x_i^*)(x_i - x_{i-1}) - \int_a^b g(x)\,dx \right| \\
&\qquad < \epsilon/2 + \epsilon/2 + \epsilon.
\end{aligned}$$

Thus by definition, $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

2. Exericse.

□

Proposition 5.11. *If $f : [a,b] \to \mathbb{R}$ is integrable, and $f(x) \geq 0$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \geq 0$.*

Proof. For any $\epsilon > 0$ we can find a Riemann sum $S = \sum_{i=1}^n f(x_i^*)(x_i - x_{i-1})$ such that $|S - \int_a^b f(x)\,dx| < \epsilon$. Then we have

$$\begin{aligned}
\epsilon &> S - \int_a^b f(x)\,dx \\
\int_a^b f(x)\,dx &> S - \epsilon.
\end{aligned}$$

But clearly $S \geq 0$, so $\int_a^b f(x)\,dx > -\epsilon$.

This is true for any $\epsilon > 0$, so we can conclude that $\int_a^b f(x)\,dx \geq 0$.

□

Corollary 5.12. *If $f, g : [a,b] \to \mathbb{R}$ are integrable, and $f(x) \leq g(x)$ for all $x \in [a,b]$, then*

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Corollary 5.13. *If $f : [a,b] \to \mathbb{R}$ is integrable, and $m, M \in \mathbb{R}$ with $m \leq f(x) \leq <$ for all $x \in [a,b]$, then*

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

5.3 Existence of Integrals

Now that we understand some properties of the integral, we would like to know when the integral is well-defined. In this section and the next we will prove results about which functions are integrable. The primary result is that any continuous function defined on a closed interval is integrable, which we will see in section 5.4. But first we will prove some other results about integrability.

We begin with a technical lemma that completely but awkwardly characterizes the integrable functions. This lemma is essentially an application of completeness from section 3.2, and is the only way we can really prove that integrals have to converge in the abstract.

Lemma 5.14. *Let $f : [a, b] \to \mathbb{R}$ be a real-valued function. Then f is integrable if and only if: for any $\epsilon > 0$ there is a $\delta > 0$ so that if S_1, S_2 are two Riemann sums corresponding to partitions of width less than δ, then $|S_1 - S_2| < \epsilon$.*

Proof. First, suppose f is integrable and $\int_a^b f(x)\,dx = I$. If $\epsilon > 0$, there exists a $\delta > 0$ so that if S is a Riemann sum corresponding to a partition of width less than δ, then $|S - I| < \epsilon/2$.

So suppose S_1, S_2 are two Riemann sums for f corresponding to partitions of width less than δ. Then

$$|S_1 - S_2| = |S_1 - I - (S_2 - I)| \leq |S_1 - I| + |S_2 - I| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Conversely, suppose $f : [a, b] \to \mathbb{R}$, and for every $\epsilon > 0$ there is a $\delta > 0$ so that whenever S_1, S_2 are Riemann sums corresponding to partitions of width less than δ, then $|S_1 - S_2| < \epsilon$. We wish to show that f is integrable.

We begin by finding a plausible integral, by constructing a sequence which is Cauchy and thus convergent. For each $n \in \mathbb{N}$, choose some partition of width $1/n$ and some Riemann sum S_n corresponding to that partition. Then (S_n) is a sequence of real numbers.

We claim that S_n is Cauchy. Let $\epsilon > 0$. Then there is a $\delta > 0$ so that whenever S_1, S_2 are Riemann sums corresponding to partitions of width less than δ, then $|S_1 - S_2| < \epsilon$. So let $N > 1/\delta$, and then if $n, m > N$ then $1/n, 1/m < \delta$ and so $|S_n - S_m| < \epsilon$.

Then (S_n) is a Cauchy sequence of real numbers, so it has a limit, which we will call I. We claim that $I = \int_a^b f(x)\,dx$. Let $\epsilon > 0$. Then there is some $\delta_1 > 0$ so that whenever S, T are Riemann sums corresponding to partitions of width less than δ_1, then $|S - T| < \epsilon/2$, and there is some $N \in \mathbb{N}$ so that if $n > N$ then $|S_n - I| < \epsilon/2$.

Let $\delta = \min\{\delta_1, 1/N\}$, and suppose S is a Riemann sum corresponding to a partition of width less than δ. If $n > N$ we know that $|S - S_n| < \epsilon/2$, and $|S_n - I| < \epsilon/2$. Then

$$|S - I| = |S - S_n + S_n - I| \leq |S - S_n| + |S_n - I| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus $I = \int_a^b f(x)\,dx$ by definition.

$\square$

Remark 5.15. This sort of completeness result really is necessary to make integrals converge. If we consider integrals in the rationals, then $\int_1^2 \frac{dx}{x}$ is perfectly well-defined and seems like the sort of thing that ought to converge. But of course it doesn't, since it would converge to $\ln(2) \notin \mathbb{Q}$.

Before we move on to continuous functions, I want to focus on another type of function whose integrals always exist, and see how much we can learn from those.

Definition 5.16. We say $f : [a,b] \to \mathbb{R}$ is a *step function* if there is some partition $P = \{x_0, \dots, x_n\}$ for $[a,b]$ so that f is constant on each open interval (x_i, x_{i+1}).

Example 5.17. Any constant function is a step function.

The function $f(x) = \begin{cases} 0 & x < 1 \\ 3 & 1 < x < 2 \\ 1 & 2 < x \end{cases}$ is a step function.

The function $\lfloor x \rfloor$ that gives the greatest integer less than or equal to x is a step function.

Clearly step functions should be integrable—the standard Riemann sum picture with rectangles is not just an approximation here as long as you choose your partition sensibly. We still need to prove that step functions are integrable, though, since the integral needs to converge for *any* partition, not just for sensible ones.

A class doing this in greater depth would define the concept of a *refinement* of a partition, which is a partition with more points that includes all the points of the starting partition. With that concept this proof is easy direclty from the definition. But there's a far easier argument that we can make using the the ideas of section 5.2.

Lemma 5.18. *Any step function is integrable.*

Further, if $f : [a,b] \to \mathbb{R}$ is a step function defined on the partition $\{x_0, \dots, x_n\}$ so that $f(x) = c_i$ for all $x_{i-1} < c_i < x_i$, then

$$\int_a^b f(x)\,dx = \sum_{i=1}^n c_i(x_i - x_{i-1}).$$

Proof. We can prove this directly, but that's a huge pain. It's much easier to use results we already have to prove this.

For each i, define $\phi_i(x) : [a,b] \to \mathbb{R}$ by $\phi_i(x) = \begin{cases} 1 & x_{i-1} < x < x_i \\ 0 & \text{otherwise} \end{cases}$ (This means that $\phi_i(x) = \chi_{(x_{i-1},x_i)}(x)$ is the characteristic function of the open interval (x_{i-1}, x_i), but that notation is extremely cumbersome). It's easy to see (by example 5.8) that $\int_a^b \phi_i(x)\,dx = (x_i - x_{i-1})$.

It's not quite the case that $f(x) = \sum_{i=1}^n c_i\phi_i(x)$, but it's close. We see that $f(x) - \sum_{i=1}^n c_i\phi_i(x) = 0$ unless $x = x_i$ for some i; thus it is zero except at finitely many points. By exercise 5.7 we know that

$$\int_a^b \left(f(x) - \sum_{i=1}^n c_i\phi_i(x) \right) dx = 0.$$

But then by linearity, we we have

$$\begin{aligned}
\int_a^b f(x)\,dx &= \int_a^b \left(f(x) - \sum_{i=1}^n c_i\phi_i(x) + \sum_{i=1}^n c_i\phi_i(x) \right) dx \\
&= \int_a^b \left(f(x) - \sum_{i=1}^n c_i\phi_i(x) \right) dx + \int_a^b \left(\sum_{i=1}^n c_i\phi_i(x) \right) dx \\
&= 0 + \sum_{i=1}^n c_i \int_a^b \phi_i(x)\,dx \\
&= \sum_{i=1}^n c_i(x_i - x_{i-1}).
\end{aligned}$$

Fact 5.19. *Let $f : [a, b] \to \mathbb{R}$. Then f is integrable if and only if for every $\epsilon > 0$ there exist step functions $f_1, f_2 : [a, b] \to \mathbb{R}$ such that*

$$f_1(x) \leq f(x) \leq f_2(x) \qquad \textit{for every } x \in [a, b]$$

and

$$\int_a^b f_2(x) - f_1(x)\,dx < \epsilon.$$

This is effectively a variation on the Squeeze Theorem. This idea eventually grows into the concept of a Lesbesgue integral, where very much like this is taken to be the definition of an integral.

□

5.4 Uniform Continuity and Integrals of Continuous Functions

The main goal of this section is to prove that a continuous function is integrable. The basic idea is this: on a small subinterval, the distance between the maximum and minimum values of x are close together—since the point of continuity is that when inputs are close together, so are outputs. So as partitions get smaller, the difference between the maximum and minimum Riemann sums will get smaller.

But this argument by itself doesn't quite work. As each subinterval gets smallerthe maximum and minimum values get closer together; but you would need to do this for each subinterval individually, and the number of subintervals goes to infinity so this doesn't actually work. We need some guarantee that we can control the function everywhere at once.

Definition 5.20. Let (E, d) and (F, d') be metric spaces, and let $f : E \to F$ be a function of metric spaces. We say f is *uniformly continuous* if, for every $\epsilon > 0$, there is a $\delta > 0$ so that if $x, y \in E$ and $d(x, y) < \delta$, then $d'(f(x), f(y)) < \epsilon$.

This definition is different from continuity in that we don't have a special point we're considering. We have to pick one δ that works everywhere.

Example 5.21. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$. We claim f is uniformly continuous.

Let $\epsilon > 0$ and let $\delta = \epsilon$. If $|x - y| < \delta$ then $|f(x) - f(y)| = |x - y| < \delta = \epsilon$.

Example 5.22. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = x^2$. We claim f is uniformly continuous.

Let $\epsilon > 0$, and set $\delta = \epsilon/2$. If $|x - y| < \delta$, then $|x + y| \leq 2$, so

$$|x^2 - y^2| = |x - y| \cdot |x + y| < \delta|x + y| \leq 2\delta = \epsilon.$$

Example 5.23. Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = 1/x$. Then f is continuous, since it's given by algebraic operations. We claim that f is not uniformly continuous. Heuristically, this is true because the same distance on the x-axis creates larger and larger distances on the y-axis as we move closer to $x = 0$.

To prove this, fix $\epsilon > 0$, and fix $\delta > 0$. We want to find x, y so that $|x - y| < \delta$ but $\left|\frac{1}{x} - \frac{1}{y}\right| \geq \epsilon$. There are lots of options here. If for simplicity we take $y = x/2$, then we need $0 < x < 2\delta$ and

$$\epsilon \leq \left|\frac{1}{x} - \frac{1}{x/2}\right| = \left|\frac{1}{x} - \frac{2}{x}\right| = \frac{1}{x},$$

which is equivalent to $x \leq 1/\epsilon$.

So if we take $x < \min\{1/\epsilon, 2\delta, 1\}$ and $y = x/2$, then we have $|x - y| < \delta$ and $|1/x - 1/y| \geq \epsilon$. Since we can find these x, y for any δ, we know that no δ exists that "works" for every x, y, and thus f is not uniformly continuous.

Exercise 5.24. *Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = \sin(1/x)$. Show that f is not uniformly continuous.*

However, the problem is, essentially, when we have to deal with infinitely many intervals. If we can somehow limit ourselves to finitely many intervals, we can make everything work out.

Proposition 5.25. *A continuous function on a compact set is uniformly continuous.*

Explicitly: let (E, d) and (F, d') be metric spaces, and $f : E \to F$ continuous. If E is compact, then f is uniformly continuous

Proof. We will use topological compactness again here. A proof using sequential compactness exists, but is much more complicated.

The basic idea is that since f is continuous on E, we can find a δ at each point. Since there are infinitely many points, we have infinitely many δ, and we can't just take the minimum one. But the point of compactness is to allow us to treat an infinite set sort of like a finite set; compactness allows us to narrow this down to finitely many δ, at which point we can take the minimum.

Let $\epsilon > 0$. We want to show that, for some $\delta > 0$, then whenever $d(x, y) < \delta$ we know that $d'(f(x), f(y)) < \epsilon$. So assume for contradiction no such δ exists.

Because f is continuous on E, for each $x \in E$, we can find a real number $\delta_x > 0$ so that if $d(x, y) < \delta_x$ then $d'(f(x), f(y)) < \epsilon/2$. Then for each x we can consider the ball of radius $\delta_x/2$ centered at x. Since $x \in B_{\delta_x/2}(x)$ for each $x \in E$, we know that $E = \bigcup_{x \in E} B_{\delta_x/2}(x)$.

We have now written E as an infinite union of open sets. Since E is compact, we can in fact choose finitely many of those sets and use them to cover all of E. So there exist $x_1, \dots, x_n \in E$ such that $E = \bigcup_{i=1}^n B_{\delta_{x_i}/2}(x_i)$. We set $\delta = \min\{\delta_{x_i}/2\}$.

Now suppose $x, y \in E$ and $d(x, y) < \delta$. We want to show that $f(x)$ and $f(y)$ are close together; we can do this by showing that x and y are both close to some x_i, and then using the definition of δ above to show that $f(x)$ and $f(y)$ are both close to $f(x_i)$.

Since $x \in E = \bigcup_{i=1}^n B_{\delta_{x_i}/2}(x_i)$, there is some i so that $x \in b_{\delta_{x_i}/2}(x_i)$, and thus $d(x, x_i) < \delta_{x_i}/2$. But we can show that y also has to be close to x_i; because

$$d(y, x_i) \leq d(y, x) + d(x, x_i) < \delta + \delta_{x_i}/2 < \delta_{x_i}/2 + \delta_{x_i}/2 = \delta_{x_i}.$$

Now we have shown that since x and y are close together, there must be some x_i they are both close to. Now we just use this to show that $f(x)$ and $f(y)$ have to be close together.

But we know that $d(f(x), f(x_i)) < \epsilon/2$ since $d(x, x_i) < \delta \leq \delta_{x_i}/2 < \delta_{x_i}$. And similarly we know that $d(y, x_i) < \delta_{x_i}$ and thus $d(f(y), f(x_i)) < \epsilon/2$. So

$$d(x, y) \leq d(x, x_i) + d(x_i, y) < \epsilon/2 + \epsilon/2 = \epsilon.$$

□

Now we can finally prove the convergence theorem for integrals of continuous functions.

Theorem 5.26. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $\int_a^b f(x)\, dx$ exists.*

Proof. Recall our Cauchy criterion for proving integrals converge: lemma 5.14 says that f is integrable if and only if: for any $\epsilon > 0$ there is a $\delta > 0$ so that if S_1, S_2 are two Riemann sums corresponding to partitions of width less than δ, then $|S_1 - S_2| < \epsilon$. We combine this with our results on uniform continuity to show that any continuous function is integrable.

Since f is continuous on the compact set $[a, b]$, we know that f is in fact uniformly continuous. Thus for any $\epsilon > 0$ we can find a $\delta > 0$ so that if $|x-y| < \delta$ then $|f(x)-f(y)| < \frac{\epsilon}{2(b-a)}$.

Let S be a Riemann sum corresponding to a partition $P = \{x_0, \dots, x_n\}$, and S' be a Riemann sum corresponding to a partition $Q = \{y_0, \dots, y_m\}$, with $\text{width}(P), \text{width}(Q) < \delta$. We want to show that $|S - S'| < \epsilon$.

Our approach is to define another Riemann sum T that corresponds to another (specific) partition. Then we will show that $|S - T|, |S' - T| < \epsilon/2$. Once we have show this, the triangle inequality whill tell us that $|S - S'| < \epsilon$, and thus by our "completeness" lemma we will know that f is integrable.

Let $P \cup Q = \{z_0, \dots, z_\ell\}$ be the partition obtained by taking all the points that are in either P or Q. Then it is clear that $\text{width}(P \cup Q) < \delta$. Let T be a Riemann sum corresponding to $P \cup Q$.

We can write S as a not-quie-Riemann sum corresponding to $P \cup Q$, by simply splitting each interval of P up into the intervals of $P \cup Q$. Then we get something like

$$\begin{aligned} S &= \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} f(x_i^*)(z_{j_i} - z_{j_{i-1}}) \\ &= \sum_{i=1}^{n} f(x_i^*) \left(\sum_{k=j_{i=1}}^{j_i} z_k - z_{k-1} \right) \\ &= \sum_{k=1}^{\ell} f(x_i^*)(z_k - z_{k-1}). \end{aligned}$$

This isn't technically a Riemann sum because x_i^* doesn't necessarily belong to every sub-subinterval.

But now, if T is any Riemann sum corresponding to $P \cup Q$ we have

$$\begin{aligned} |S - T| &= \left| \sum_{k=1}^{\ell} f(x_i^*)(z_k - z_{k-1}) \right| - \left| \sum_{k=1}^{\ell} f(z_k^*)(z_k - z_{k-1}) \right| \\ &\leq \left| \sum_{k=1}^{\ell} \left(f(x_i^*) - f(z_k^*) \right) (z_k - z_{k-1}) \right| \\ &\leq \sum_{k=1}^{\ell} \left| f(x_i^*) - f(z_k^*) \right| (z_k - z_{k-1}). \end{aligned}$$

But since x_i^* and z_k^* are in the same subinterval of the partition P, and we know width $P < \delta$, then by uniform continuity we know that $|f(x_i^*) - f(z_k^*)| < \frac{\epsilon}{2(b-a)}$. Thus

$$\begin{aligned} |S - T| &\leq \sum_{k=1}^{\ell} \frac{\epsilon}{2(b-a)}(z_k - z_{k-1}) \\ &= \frac{\epsilon}{2(b-a)} \sum_{k=1}^{\ell}(z_k - z_{k-1}) \\ &= \frac{\epsilon}{2(b-a)}(b-a) = \epsilon/2. \end{aligned}$$

Nothing in this argument depended on the specific properties of S and T, so by the exact same argument we can conclude that $|S' - T| < \epsilon/2$. Thus the triangle inequality tells us that $|S - S'| < \epsilon$. We have shown that for any $\epsilon > 0$ there is a $\delta > 0$ such that if S, S' are two Riemann sums corresponding to partitions of width less than δ, then $|S - S'| < \epsilon$. Thus by lemma 5.14, we know that f is integrable.

□

5.5 The Fundamental Theorem of Calculus

In this section we want to use the integral to define new functions. In particular, if $f : [c, d] \to \mathbb{R}$ is integrable, and $a \in [c, d]$, then we can define a function $F(x) = \int_a^x f(t)\, dt$. Since f is integrable, this function is well-defined for any $x \neq a$, and if we define $\int_a^a f(t)\, dt = 0$ then F is defined on all of $[c, d]$. We call F an *indefinite integral* of f.

In fact F is differentiable, and we want to compute the derivative of F. But we need a couple intermediate results first.

Proposition 5.27. *If $a < b < c$ and $f : [a, c] \to \mathbb{R}$ is a function, then f is integrable on $[a, c]$ if and only if it is integrable on $[a, b]$ and $[b, c]$, and if it is integrable, we have the equality*

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx.$$

Remark 5.28. If we define $\int_a^a f(x)\, dx = 0$ and $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$, then this result holds for any a, b, c such that at least two of the integrals exist.

Theorem 5.29 (Fundamental Theorem of Calculus)**.** *Let $U \subset \mathbb{R}$ be an open interval with $a \in U$, and let $f : U \to \mathbb{R}$ be continuous. Define $F : U \to \mathbb{R}$ by $F(x) = \int_a^x f(t)\, dt$. Then F is differentiable, and $F'(x) = f(x)$.*

Proof. Since f is continuous, we know that $F(x)$ is defined for any $x \in U$. W need to compute the derivative. So we have

$$\begin{aligned} F'(x) &= \lim_{h \to 0} \frac{1}{h} \left(\int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt \right) \\ &= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt. \end{aligned}$$

We wish to show that this limit is equal to $f(x)$. We compute that

$$\begin{aligned} \left| \frac{1}{h} \int_x^{x+h} f(t)\, dt - f(x) \right| &= \left| \frac{1}{h} \int_x^{x+h} f(t)\, dt - \frac{1}{h} \int_x^{x+h} f(x)\, dt \right| \\ &= \left| \frac{1}{h} \int_x^{x+h} f(t) - f(x)\, dt \right| \\ &\leq \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\, dt. \end{aligned}$$

Since f is continuous at x, for any $\epsilon > 0$ we can find a δ so that if $|t - x| < \delta$ then $|f(t) - f(x)| < \epsilon$. Suppose $|h| < 0$. Then for any $t \in (x, x+h)$ we know that $|f(t) - f(x)| < \epsilon$, and so

$$\frac{1}{h}\int_x^{x+h} |f(t)-f(x)|\,dt < \frac{1}{h}\int_x^{x+h} \epsilon\,dt$$
$$= \frac{1}{h}h\epsilon = \epsilon.$$

Thus by definition, $F'(x) = f(x)$.

□

Corollary 5.30. *If $f : (c,d) \to \mathbb{R}$ is continuous, then there is a differentiable function $F : (c,d) \to \mathbb{R}$ such that $F'(x) = f(x)$.*

Proof. If $a \in (c,d)$, define $F(x) = \int_a^x f(t)\,dt$. □

Corollary 5.31. *If $F : (c,d) \to \mathbb{R}$ is differentiable and $F'(x) = f(x)$, and $[a,b] \subset (c,d)$, then $\int_a^b f(t)\,dt = F(b) - F(a)$.*

Proof. We know that

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt - F(x)\right) = f(x) - f(x) = 0.$$

Thus $\int_a^x f(t)\,dt = F(x) + C$ for some fixed constant $C \in \mathbb{R}$. We can find our constant by plugging in a for x, so we get

$$F(a) + C = \int_a^a f(t)\,dt = 0$$

and thus $C = -F(a)$. So

$$\int_a^x f(t)\,dt = F(x) - F(a)$$
$$\int_a^b f(t)\,dt = F(b) - F(a).$$

□

Proof of Proposition 5.27. □

Corollary 5.32 (Change of Variables)**.** *Let $U, V \subset \mathbb{R}$ be open intervals, $\phi : U \to V$ be continuously differentiable, and $f : V \to \mathbb{R}$ continuous. Then*

$$\int_{\phi(a)}^{\phi(b)} f(v)\,dv = \int_a^b f(\phi(u))\phi'(u)\,du.$$

Proof. Define $F : V \to \mathbb{R}$ by $F(y) = \int_{\phi(a)}^{y} f(v)\,dv$. Then $F'(y) = f(y)$. Then if we define $G : U \to \mathbb{R}$ by $G(x) = \int_{\phi(a)}^{\phi(x)} f(v)\,dv$, we see that $G(x) = F(\phi(x))$, and then by the chain rule we have

$$G'(x) = F'(\phi(x))\phi'(x) = f(\phi(x))\phi'(x).$$

But then G is an antiderivative for $f(\phi(x))\phi'(x)$, and thus

$$G(x) = \int_a^x f(\phi(u))\phi'(u)\,du + C$$

for some constant $C \in \mathbb{R}$. But we can compute that

$$G(a) = \int_{\phi(a)}^{\phi(a)} f(v)\,dv = 0$$

$$\int_a^a f(\phi(u))\phi'(u)\,du = 0$$

and thus $0 = G(a) = 0 + C$ and so $C = 0$. Thus

$$G(x) = \int_a^x f(\phi(u))\phi'(u))\,du.$$

□

5.6 Logarithm and Exponent

Definition 5.33. Define $\log(x) = \int_1^x \frac{dt}{t}$.

Proposition 5.34. *The function* $\log(x)$ *is a differentiable function on* $(0, \infty)$ *with* $\frac{d}{dx}\log(x) = \frac{1}{x}$. *It is strictly increasing,* $\log(1) = 0$ *and has image all of* $\mathbb{R}$*, and*

- $\log(xy) = \log(x) + \log(y)$
- $\log(x/y) = \log(x) - \log(y)$
- $\log(x^n) = n\log(x)$ *for* $n \in \mathbb{Z}_{\geq 0}$.

Proposition 5.35. *The derivative result follows directly from the definition and the fundamental theorem of calculus. It is increasing since* $\frac{d}{dx}\log(x) = \frac{1}{x} > 0$. *We know* $\log(1) = \int_1^1 \frac{dt}{t} = 0$.

Let $z = xy$. *Then*

$$\frac{d}{dx}\log(z) = \frac{1}{z} \cdot \frac{d}{dx}z = \frac{1}{xy} \cdot y = \frac{1}{x}.$$

This means that $\log(z) = \log(x) + C$ *for some* $C \in \mathbb{R}$. *Plugging in* $x = 1$ *gives us* $\log(y) = 0 + C$, *and thus* $\log(xy) = \log(x) + \log(y)$.

If $x = 1/y$ *then we have* $\log(1) = \log(y) + \log(1/y)$ *and thus* $\log(1/y) = -\log(y)$*; then*

$$\log(x/y) = \log(x \cdot 1/y) = \log(x) + \log(1/y) = \log(x) - \log(y).$$

The last result follows by induction on n.