6 Sequences of Functions

Definition 6.1. For each $n \in \mathbb{N}$, let $f_n : E \to F$ be a function of metric spaces. Then for each a we have a sequence $f_n(a)$ of points in F. If $f : E \to F$, we say that f_n *converges pointwise* to f if $\lim_{n\to\infty} f_n(a) = f(a)$ for each $a \in E$. We write $f = \lim_{n\to\infty} f_n$.

Example 6.2. The sequence of functions $f_n(x) = \frac{x}{n}$ converges pointwise to the contant zero function.

Example 6.3. Let $f_n(x) = \frac{1}{n}x + \left(1 - \frac{1}{n}\right) x^2$. Then $\lim_{n\to\infty} f_n = f$ where $f(x) = x^2$.

We'd like to be able to extend properties of our sequences to their limits. For example, we showed that the limit of a sequence of positive numbers is positive. The most important property of functions is continuity. Is the limit of a sequence of continuous functions continuous?

Example 6.4. Let $f_n : [0, 1] \to \mathbb{R}$ be defined by $f_n(x) = x^n$. Then for each $x \in [0, 1)$, we have $\lim_{n\to\infty} x^n = 0$, but $\lim_{n\to\infty} 1^n = 1$. So $\lim_{n\to\infty} f_n = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$

The assertion that the limit of a sequence of continuous functions is sometimes called "Cauchy's Wrong Theorem". The basic problem is that while each function in the sequence is continuous, the functions can get steeper and steeper as we get further into the sequence; each function has finite steepness and is continuous, but in the limit this steepness goes to infinity.

But this can only happen in this example because, while $f_n(x)$ goes to zero for every $x \neq 1$, it does this slower and slower as x gets closer to 1, allowing the function to stretch. If we find a way to prevent that, we can keep the function from stretching and becoming discontinuous in the limit.

Definition 6.5. For each n, let $f_n : E \to F$ be a function of metric spaces. We say that the sequence of functions (f_n) *converges uniformtly* to a function f if, for any $\epsilon > 0$, there is a $N \in \mathbb{N}$ so that if $n > N$, then $d(f_n(x), f(x)) < \epsilon$ for any $x \in E$.

Notice we've seen an idea like this before. A function is continuous if we can find a δ for any pair of ϵ and x; it is uniformly continuous if for each ϵ, we can find a δ that works for any x. Similarly, functions converge pointwise if for each x and ϵ you can find an N that works; they converge uniformly if you can find an N that works for any x.

Clearly, a sequence that converges uniformly also converges pointwise.

Example 6.6. If $f_n : \mathbb{R} \to \mathbb{R}$ is given by $f_n(x) = x/n$, then the convergence is not uniform; for any $\epsilon > 0, N \in \mathbb{N}$, we can take $x > (N+1)\epsilon$ and then $|f_{N+1}(x) - 0| = x/(N+1) > \epsilon$.

However, if we define $f_n : [0,1] \to \mathbb{R}$ instead, then the sequence converges uniformly. Let $\epsilon > 0$ and set $N > 1/\epsilon$. Then if $n < N$ we have

$$|f_n(x) - 0| = |x/n| < x/N < \epsilon x \leq \epsilon$$

since $0 \leq x \leq 1$.

Example 6.7. Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = x^n$. Then f_n converges to $f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$ pointwise, but the convergence is not uniform.

Let $\epsilon = 1/2$, and let $N \in \mathbb{N}$. We want to show that there is some $n > N$ and some $x \in [0,1]$ so that $d(f_n(x), f(x)) \geq \epsilon = 1/2$.

For each $x \neq 1$ we have $f(x) = 0$, so we just want to show that $f_n(x) \geq 1/2$. This is equivalent to $x^n \geq 1/2$ or $x \geq \sqrt[n]{1/2}$. So let $n < N$ and $x = \sqrt[n]{1/2}$. Then $d(f_n(x), f(x)) = |1/2 - 0| = 1/2 \geq \epsilon$.

Thus we have no N so that if $n > N$ then $d(f_n(x), f(x)) < \epsilon$ for any x; thus f_n does not converge to f uniformly.

Exercise 6.8. *If f_n converges to f uniformly, then f_n converges to f pointwise.*

Proposition 6.9. *Suppose $f_n : E \to F$ is a sequence of functions each continuous at some point $a \in E$, and f_n converges uniformly to some function $f : E \to F$. Then f is continuous at a.*

Proof. We want to show that f is continuous at a. This means that for any $\epsilon > 0$, there is some $\delta > 0$ so that if $d(x,a) < \delta$ then $d'(f(x), f(a)) < \epsilon$.

Since f_n converges to f uniformly, there is some $N \in \mathbb{N}$ so that if $n > N$, then $d'(f_n(x), f(x)) < \epsilon/3$ for any $x \in E$. So let $n < N$.

We know that f_n is continuous. So there is some δ so that if $d(x,a) < \delta$ then $d'(f_n(x), f_n(a)) < \epsilon/3$. But then we have

$$\begin{aligned} d'(f(x), f(a)) &\leq d'(f(x), f_n(x)) + d'(f_n(x), f_n(a)) + d'(f_n(a), f(a)) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

□

This idea of uniform convergence is really powerful. With pointwise convergence, we can make f_n work at any given point for large enough n, but we can't necessarily make things work for every point at the same time. With uniform convergence, we guarantee that we can make things work for the entire function at once.

(Recall we've seen this same effect when we talked about uniform continuity: it made a big difference that we could be continuous everywhere with the same δ).

But also, we've seen what's effectively uniform continuity before, in a completely different context.

Definition 6.10. Let $\mathcal{B}(E, F)$ be the set of bounded functions $f : E \to F$. We can define the sup metric by $d_\infty(f, g) = \sup\{d'(f(x), g(x)) : x \in E\}$.

Thus a sequence f_n converges to a function f uniformly if and only if it converges in the sup metric.

Proposition 6.11. *If F is complete, then $\mathcal{B}(E, F)$ is complete.*

Proof. Let f_n be a Cauchy sequence of functions. Then for each $\epsilon > 0$ there is an $N \in \mathbb{N}$ so that if $n, m > N$ then $d_\infty(f_n, f_m) < \epsilon$. We want to define a function f so that $\lim_{n\to\infty} f_n = f$.

Let $x \in E$. Then we have a sequence $(f_n(x)) \subset F$. For any $\epsilon > 0$, there is a $N \in N$ so that if $n, m > N$, then $d_\infty(f_n, f_m) < \epsilon$. Then

$$d'(f_n(x), f_m(x)) \leq \sup\{d'(f_n(x), f_m(x)) : x \in E\} < \epsilon.$$

Thus the sequence $(f_n(x))$ is Cauchy, and since it is a Cauchy sequence in the complete metric space F, it has a limit. So for each x, define $f(x) = \lim_{n\to\infty} f_n(x)$.

Now we just need to show that $\lim_{n\to\infty} f_n = f$ in the sup metric. Fix $\epsilon > 0$. There is a $N \in \mathbb{N}$ so that if $n, m > N$ then $d_\infty(f_n, f_m) < \epsilon/2$. If we fix $n > N$, then for all $m > N$ we have $d'(f_n(x), f_m(x)) < \epsilon/2$, and thus $f_m(x) \in \overline{B}(f_n(x))$. Thus the sequence $(f_m(x))$ is eventually contained in $\overline{B}(f_n(x))$, and since the closed ball is closed, the limit $\lim_{m\to\infty} f_m(x) \in \overline{B}(f_n(x))$.

Thus $d'(f(x), f_n(x)) \leq \epsilon/2$, and for any $m > N$ we have

$$d'(f_m(x), f(x)) \leq d'(f_m(x), f_n(x)) + d'(f_n(x), f(x)) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus $\lim_{n\to\infty} f_n = f$ in the sup metric. So every Cauchy sequence converges, and $\mathcal{B}(E, F)$ is complete. □

When we work with this space at an advanced level, we make a minor technical tweak where we treat functions as equivalent if they differ at only finitely many points, and define the metric as the supremum ignoring finitely many points. (Even more technically, we ignore functions that differ on a "measure zero" set). But everything we said is true without worrying about that.

However, we can ignore all of this by specifying a bit. It's not obvious, but it is true, that if two *continuous* functions differ at only finitely many points, they are in fact identical. So if we need to take a specific representative of this equivalence class, and taking a continuous function is an option, we do that. In fact, we can say a bit more about this specific case:

Corollary 6.12. *If E is compact and F is complete, then $\mathcal{C}(E, F)$ is a complete metric space under the sup metric.*

Proof. Since E is compact, every continuous function is bounded. And by proposition 6.9, the limit of any continuous function is continuous. Thus $\mathcal{C}(E, F)$ is a closed subset of a complete metric space, and thus complete. □

Remark 6.13. There are other ways to talk about function convergence. Earlier in the course we talked about $L^1([a, b], \mathbb{R})$ with the metric $d_1(f, g) = \int_a^b |f(x) - g(x)|\, dx$. In fact, for any $p \geq 1$ we can define

$$d_p(f, g) = \sqrt[p]{\int_a^b |f(x) - g(x)|^p\, dx}.$$

These different metrics are analogous to the various metrics we defined on $\mathbb{R}^n$; but unlike in the case of $\mathbb{R}^n$, they are actually genuinely distict metrics with different convergence properties.

Understanding the L^p spaces is *far* beyond the scope of this course; I'll only say that we genuinely can't understand them fully without using the equivalence class idea I mentioned earlier, since otherwise we lose non-negativity of metrics. (A function that is zero except at one point has integral zero).

But the theory of what each of these different metrics does to convergence is deep and interesting, and useful in fields like statistics and differential equations.