Math 310 Fall 2018
Real Analysis HW 10 Solutions
Due Monday, November 19

For all of these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. Prove that $\int_0^1 x\,dx = 1/2$ directly from the definition of Riemann integral. (You may assume that the integral exists if you wish).

   Solution: We will do this in two steps. First we'll show that we can find a sequence of Riemann sums that converges to 1/2. Then we'll show that if the integral exists, this implies it must be 1/2.

   For each n, define a partition $P_n = \{0, 1/n, \ldots, n/n\}$, and a corresponding Riemann sum

$$\begin{aligned} S_n &= \sum_{i=1}^{n} f(i/n)(i/n - (i-1)/n) \\ &= \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} i \\ &= \frac{1}{n^2} \frac{n(n-1)}{2} = \frac{1 - 1/n}{2}. \end{aligned}$$

   It's easy to see that $\lim_{n\to\infty} S_n = 1/2$.

   We know the integral exists, so set $I = \int_0^1 x\,dx$. Then for any $\epsilon > 0$, we can find a δ so that if S is a Riemann sum for a partition of width less than δ, then $|S - I| < \epsilon/2$. But we can also find an N' so that if $n > N'$ then $|S_n - 1/2| < \epsilon/2$. Let $N > \max\{N', 1/\delta\}$.

   Then if $n > N$, we know that $|S_n - 1/2| < \epsilon/2$. But also we have $1/n < \delta$, so the width of P_n is less than δ, so we know that $|S_n - I| < \epsilon/2$. By the triangle inequality, we have $|1/2 - I| < \epsilon$.

   But this holds for any $\epsilon > 0$, so $|1/2 - I|$ is less than any positive number. Thus $|1/2 - I| = 0$ and $\int_0^1 x\,dx = I = 1/2$.

2. Let $c \in (a, b)$, and let $f : [a, b] \to \mathbb{R}$ be defined by $f(c) = 1$ and $f(x) = 0$ if $x \neq c$. Prove that $\int_a^b f(x)\,dx = 0$.

   Solution: Fix $\epsilon > 0$, and let $\delta = \epsilon/2$. Let $P = \{x_0, \ldots, x_n\}$ be a partition of width less than δ. Then any Riemann sum corresponding to P can be written

$$S = \sum_{i=1}^{n} f(x_i^*)(x_i - x_i - 1).$$

But $x_i^* \neq c$ except for possibly two times, since c can appear in at most two of these intervals. (It appears in two if it is in fact one of the x_i). Thus we have

$$\begin{aligned} S &= 0 + f(x_i^*)(x_i - x_{i-1}) + f(x_{i+1}^*)(x_{i+1} - x_i) \\ |S - 0| &= |f(x_i^*)(x_i - x_{i-1}) + f(x_{i+1}^*)(x_{i+1} - x_i)| \\ &\leq |f(x_i)^*|\delta + |f(x_{i+1}^*)|\delta \\ &< 2\delta = \epsilon. \end{aligned}$$

Thus by definition, $\int_a^b f(x)\,dx = 0$.

3. Let $f : [a, b] \to \mathbb{R}$ such that $\int_a^b f(x)\,dx$ exists. Prove that $\int_{a+c}^{b+c} f(x-c)\,dx$ exists, and is equal to $\int_a^b f(x)\,dx$.

   Solution: Fix $\epsilon > 0$. Then since $\int_a^b f(x)\,dx$ exists, there is some δ so that if S is any Riemann sum corresponding to a partition of $[a, b]$ of width less than δ, then $|S - \int_a^b f(x)\,dx| < \epsilon$.

   Now let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a + c, b + c]$ of width less than δ, and let S be a Riemann sum for $f(x - c$ corresponding to P. Then

$$\begin{aligned} S &= \sum_{i=1}^{n} f(x_i^* - c)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} f(x_i^* - c)\left((x_i - c) - (x_{i-1} - c)\right). \end{aligned}$$

   But $\{x_0 - c, \ldots, x_n - c\}$ is a partition of $[a, b]$ of width less than δ; and for each i, since $x_i^* \in [x_{i-1}, x_i]$, we know that $x_i^* - c \in [x_{i-1} - c, x_i - c]$.

   Thus S is a Riemann sum for $f(x)$ corresponding to the partition $\{x_0 - c, \ldots, x_n - c\}$, and thus we know that $|S - \int_a^b f(x)\,dx| < \epsilon$. So by definition, we know that $\int_{a+c}^{b+c} f(x - c)\,dx = \int_a^b f(x)\,dx$.

4. f $f : [a, b] \to \mathbb{R}$ is an integrable function and $c \in \mathbb{R}$, prove that $cf(x)$ is integrable and

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

   Solution: Suppose $c = 0$. Then $\int_a^b cf(x)\,dx = \int_a^b 0\,dx$ and $c\int_a^b f(x)\,dx = 0\int_a^b f(x)\,dx = 0$, so the result holds.

   Now assume $c \neq 0$. Fix $\epsilon > 0$; there is a $\delta > 0$ so that if S is a Riemann sum for f corresponding to a partition of width less than δ, then $|S - \int_a^b f(x)\,dx| < \epsilon/|c|$.

   Now suppose $P = \{x_0, \ldots, x_n\}$ is a partition of $[a, b]$ of width less than δ, and we have a Riemann sum for $cf(x)$ corresponding to this partition

$$\begin{aligned} S &= \sum_{i=1}^{n} cf(x_i^*)(x_i - x_{i-1}) \\ &= c\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}). \end{aligned}$$

Then we compute

$$\begin{aligned}
\left|S - c\int_a^b f(x)\,dx\right| &= \left|c\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) - c\int_a^b f(x)\,dx\right| \\
&= c\left|\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) - \int_a^b f(x)\,dx\right| \\
&< c\frac{\epsilon}{c} = \epsilon.
\end{aligned}$$

Thus by definition $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.