Math 310 Fall 2018
Real Analysis HW 11 Solutions
Due Monday, December 3

For all these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. Suppose $f : [a, b] \to \mathbb{R}$ is integrable. Show that f is bounded.

   Solution:

   Suppose f is integrable and let $I = \int_a^b f(x)\,dx$. Then there is some $\delta > 0$ so that if P is a partition of width less than δ and S is a Riemann sum corresponding to P, then $|S - I| < \epsilon$.

   Fix some partition $P = \{x_0, \dots, x_n\}$ of width less than δ, and fix some $1 \le j \le n$. For each $i \ne j$ fix a specific point $x_i^* \in [x_{i-1}, x_i]$. Then for every $x_j^* \in [x_{j-1}, x_j]$ we have

$$\begin{aligned}\epsilon > |S - I| &= \left|\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})\right| \\ &\ge |f(x_j^*)(x_j - x_{j-1})| - \left|\sum_{i=1}^{j-1} f(x_i^*)(x_i - x_{i-1}) + \sum_{i=j+1}^{n} f(x_i^*)(x_i - x_{i-1})\right|\end{aligned}$$

   and thus we conclude that

$$|f(x_j^*)| < \frac{1}{x_j - x_{j-1}}\left(\epsilon + \left|\sum_{i=1}^{j-1} f(x_i^*)(x_i - x_{i-1}) + \sum_{i=j+1}^{n} f(x_i^*)(x_i - x_{i-1})\right|\right).$$

   Thus the set $f([x_{j-1}, x_j])$ is bounded above by some constant C_j.

   But this argument didn't depend on j, so for each j and each $x_j^* \in [x_j, x_{j+1}]$, we can show that $|f(x_j^*)| < C_j$. Then for every $x \in [a, b]$, $f(x) < \max_{1 \le j \le n}\{C_j\}$. Thus f is bounded.

2. Prove directly from the definition that $f : (2, 4) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is uniformly continuous.

   Solution: Let $\epsilon > 0$ and set $\delta = 4\epsilon$. Then if $|x - y| < \delta$, we know that $xy > 2 \cdot 2 = 4$ and so

$$\begin{aligned}|f(x) - f(y)| = |1/x - 1/y| &= \frac{|y - x|}{|xy|} \\ &= \frac{|y - x|}{xy} < \frac{\delta}{xy} \\ &< \delta/4 = \epsilon\end{aligned}$$

Thus f is uniformly continuous.

3. Let $f : (0,1) \to \mathbb{R}$ be defined by $f(x) = \sin(1/x)$. Show that f is not uniformly continuous.

   Solution: Let $\epsilon = 1$ and let $\delta > 0$. We want to find $x, y \in (0,1)$ so that $|x - y| < \delta$ but $|f(x) - f(y)| \geq 1$.

   We want to find an $n \in \mathbb{N}$ so that

$$\begin{aligned} \frac{1}{\pi/2 + 2n\pi} &< \delta/2 \\ \pi/2 + 2n\pi &> 2/\delta \\ 2n\pi &> 2/\delta - \pi/2 \\ n &> \frac{1}{\pi\delta} - \frac{1}{4} \end{aligned}$$

   and we know such an n exists by the Archimedean property. Set $x = \frac{1}{\pi/2+2n\pi}$, and we have $f(x) = \sin(1/x) = \sin(\pi/2 + 2n\pi) = 1$.

   Now take $y = \frac{1}{3\pi/2+2n\pi}$. It's easy to see that $f(y) = \sin(1/y) = \sin(3\pi/2 + 2n\pi) = -1$. Then $|f(x) - f(y)| = |1 - (-1)| = 2 > 1$. So we just need to show that $|x - y| < \delta$. But we know that $x, y \in (0, \delta/2)$, so $|x| < \delta/2$ and $|y| < \delta/2$. Thus $|x - y| \leq |x| + |y| < \delta$.

   (If you just made $x, y \in (0, \delta)$ and observed that the distance has to be less than δ because they're in the same interval, I'll accept that too).

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Show that f is not uniformly continuous.

   Solution: Let $\epsilon > 0$ and let $\delta > 0$. We want to find $x, y \in \mathbb{R}$ with $|x - y| < \delta$ but $|f(x) - f(y)| > \epsilon$. (It's enough to show this for some specific epsilon, but I'll do it in the general case).

   Let's assume that $y = x - \delta/2$. Then $|x - y| = \delta/2$. We have

$$\begin{aligned} |f(x) - f(y)| &= |x^2 - (x - \delta/2)^2| = |x^2 - x^2 + \delta x - \delta^2/4| = |\delta x - \delta^2/4| \\ &\geq \delta x - \delta^2/4. \end{aligned}$$

   So we want to choose x so that $\delta x - \delta^2/4 \geq \epsilon$. This is the same as wanting $x \geq \frac{\epsilon - \delta^2/4}{\delta}$.

   But there is such a number, and we see that if $x \geq \frac{4\epsilon - \delta^2}{4\delta}$ and $y = x - \delta/2$, then $|x - y| = \delta/2 < \delta$ but $|f(x) - f(y)| \geq \epsilon$. Thus f is not uniformly continuous.

5. Let $f : [a,b] \to \mathbb{R}$ be continuous, such that $f(x) \geq 0$ for every $x \in [a,b]$. Suppose $c \in [a,b]$ with $f(c) > 0$. Prove that $\int_a^b f(x)\,dx > 0$.

   (Note: this result is not true if f is not continuous. See e.g. HW 10 number 2.)

   Solution: Since f is continuous, there is some $\delta > 0$ so that if $|x - c| < \delta$ then $|f(x) - f(c)| < f(c)/2$. This implies that $f(c) - f(x) < f(c)/2$ and thus $f(x) > f(c)/2$. So for all $x \in (c - \delta, c + \delta)$ we have $f(x) > f(c)/2$.

   Define $g : [a,b] \to \mathbb{R}$ by $g(x) = \begin{cases} f(c)/2 & |x - c| < \delta \\ 0 & \text{otherwise} \end{cases}$. Then we know from class that $\int_a^b g(x)\,dx = 2\delta f(c)/2 > 0$. But we also see that $f(x) \geq g(x)$ for every x, and thus from class we know that $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$. Thus

$$\int_a^b f(x)\,dx \geq \delta f(c) > 0.$$

6. Use Fact 5.19 to prove that if $f : [a,b] \to \mathbb{R}$ is increasing, then $\int_a^b f(x)\,dx$ exists.

   Solution: To prove that f is integrable, we have to prove that for every $\epsilon > 0$ we can find step functions $f_1, f_2 : [a,b] \to \mathbb{R}$ such that $f_1(x) \leq f(x) \leq f_2(x)$ for all $x \in \mathbb{R}$, and $\int_a^b f_2(x) - f_1(x)\,dx < \epsilon$.

   So fix $\epsilon > 0$. Because f is increasing, it has a maximum at b and a minimum at a. We can choose a partition $a = x_0, x_1, \ldots, x_n = b$ so that $f(x_i) - f(x_{i-1}) < \frac{\epsilon}{(b-a)}$.

   Then define f_1 as a step function for this partition so that if $x \in (x_i, x_{i+1})$ then $f(x) = f(x_i)$, and define f_2 so that if $x \in [x_i, x_{i+1})$ then $f_2(x) = f(x_{i+1})$.

   Since f is increasing, we have $f(x_i) \leq f(x) \leq f(x_{i+1})$, so for each x we have $f_1(x) \leq f(x) \leq f_2(x)$. So we just need to prove that $int_a^b f_2(x) - f_1(x)\,dx < \epsilon$. But we have

$$\begin{aligned}
\int_a^b f_2(x) - f_1(x)\,dx &= \sum_{i=1}^{n} \left(f_2(x_i^*) - f_2(x_i^*)\right)(x_{i+1} - x_i) \\
&= \sum_{i=1}^{n} \left(f(x_{i+1}) - f(x_i)\right)(x_{i+1} - x_i) \\
&< \sum_{i=1}^{n} \frac{\epsilon}{b-a}(x_{i+1} - x_i) = \frac{\epsilon}{b-a}\sum_{i=1}^{n} x_{i+1} - x_i = \epsilon.
\end{aligned}$$