Math 310 Fall 2018
Real Analysis HW 1 Solutions
Due Friday, September 7

You may *not* discuss the starred problem with classmates, though you should of course feel free to discuss it with me as much as you like. Linguistic precision is important for this problem. Submit this problem on a separate, detached sheet of paper.

$\star$ **Redo Problem:** Let F be an ordered field, and $x, y, z \in F$. Prove that if $x < 0$ and $y < z$ then $xy > xz$.

For the remainder of these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. Let F be a field and let $x_1, \ldots, x_n \in F$. Prove that the value of the expression $x_1 + x_2 + \cdots + x_n$ does not depend on the order of the terms, and any reordering will give the same value. (Hint: Use induction on the number of terms, and use the commutative property as your base case).

   Solution: We prove this with induction.

   As a base case, if $n = 2$ then $x_1 + x_2 = x_2 + x_1$ by the commutative property. These are the only possible orders for a sum of two items.

   Now assume for induction that the value of $x_1 + x_2 + \cdots + x_k$ does not depend on the order in which we write the terms for any $k \leq n$. Consider a sum $x_1 + \cdots + x_{n+1}$, and suppose we have some other order $x_{i_1} + \cdots + x_{i+n}$. We wish to show these two sums are equal.

   Suppose $i_1 = n + 1$. Then we have

   $$\begin{aligned} x_1 + \cdots + x_{n+1} &= (x_1 + \cdots + x_n) + x_{n+1} && \text{Associativity} \\ &= x_{n+1} + (x_1 + \ldots x_n) && \text{Commutativity} \\ &= x_{i_1} + (x_1 + \ldots x_n). \end{aligned}$$

   On the other hand, if $i_1 = k \neq n + 1$ then we have

   $$\begin{aligned} x_1 + \cdots + x_{n+1} &= (x_1 + \cdots + x_n) + x_{n+1} && \text{Associativity} \\ &= (x_{i_1} + x_1 + \cdots + x_{k-1} + x_{k+1} + \ldots x_n) + x_{n+1} && \text{Inductive hypothesis} \end{aligned}$$

   since the sum $(x_1 + \cdots + x_n)$ has n terms.

Either way, we see that $x_1 + \cdots + x_{n+1}$ is equal to a sum with x_{i_1} first, and all the other terms in some order second: in general, this looks like

$$x_1 + \cdots + x_{n+1} = x_{i_1} + (x_1 + \cdots + x_{k-1} + x_{k+1} + \cdots + x_{n+1}).$$

But the second term here is a sum containing only n terms, and so by our inductive hypothesis it doesn't matter the order we write the terms. Thus we conclude

$$x_1 + \cdots + x_{n+1} = x_{i_1} + \cdots + x_{i_{n+1}}$$

as desired.

Thus, by induction, the sum of any number of terms doesn't depend on the order in which they are written.

2. Prove that if F is a field and $x \in F$, then $0x = 0$.

   Solution:

$$\begin{aligned} 0x &= (0+0)x && \text{Identity} \\ &= 0x + 0x && \text{Distributivity} \\ 0x + (-0x) &= (0x + 0x) + (-0x) && \text{Inverses} \\ 0 &= 0x + (0x + -(0x)) && \text{Inverses and Associativity} \\ &= 0x + 0 && \text{Inverses} \\ &= 0x && \text{Identity} \end{aligned}$$

3. Let F be an ordered field, and let $x, y \in F$ with $0 < x < y$. Prove that $x^2 < y^2$.

   Solution:

   Since $x > 0$, by (2) from our big proposition, we know that $0 < x^2 < xy$. And since $y > 0$, by (2) we know that $0 < xy < y^2$. Then by transitivity, since $x^2 < xy$ and $xy < y^2$ we see that $x^2 < y^2$.

4. Let F be an ordered field and let $x, y, z, w \in F$. If $x \leq y$, $z \leq w$, prove that $x+z \leq y+w$. (Note: you may need to consider multiple cases here).

   Solution: If $x = y$ and $z = w$ then $x + z = y + w$.

   If $x = y$ and $z < w$, then by order additivity $x + z < y + w$. Similarly, if $x < y$ and $z = w$ then by order additivyt $x + z < y + w$.

   So suppose $x < y$ and $z < w$. Since $x < y$, by order additivity, we have $x + z < y + z$.

   Since $z < w$, by order additivity, we have $z + y < w + y$. By commutativity, this is the same as $y + z < y + w$.

   By transitivity, $x + z < y + w$.

   Thus in all cases, $x + z \leq y + w$.

5. Prove that $\mathbb{C}$ cannot be an ordered field. (Hint: is $i > 0$, $i = 0$, or $i < 0$?)

   Solution:

   Clearly $i \neq 0$. Then by (4) from our big proposition, $i^2 > 0$. But $i^2 = 0$, so we have $-1 > 0$. By (1) we then have $1 < 0$, which contradicts (5).

6. For $a, b \in \mathbb{R}$, show that

$$
\begin{aligned}
\max\{a,b\} &= \frac{a+b+|a-b|}{2} \\
\min\{a,b\} &= \frac{a+b-|a-b|}{2}
\end{aligned}
$$

Solution: Suppose $a \geq b$. Then $\max\{a,b\} = a$ and $\min\{a,b\} = b$.

Since $a \geq b$, by order additivity we have $a - b \geq b - b = 0$, so $|a-b| = a - b$. Then

$$
\begin{aligned}
\frac{a+b+|a-b|}{2} &= \frac{a+b+a-b}{2} \\
&= \frac{a+a}{2} \\
&= \frac{2a}{2} && \text{distributivity} \\
&= a && \text{inverses} \\
&= \max\{a,b\}\frac{a+b-|a-b|}{2} && = \frac{a+b-(a-b)}{2} \\
&= \frac{a-b+b+b-(a-b)}{2} \\
&= \frac{b+b}{2} \\
&= \frac{2b}{2} && \text{distributivity} \\
&= b && \text{inverses} \\
&= \min\{a,b\}
\end{aligned}
$$

If $a < b$ then we can make exactly the same argument, except $|a-b| = b - a$.

7. Let F be an ordered field and $S \subset F$. Let y be a least upper bound of S, and let $x < y$. Prove that there is an $s \in S$ such that $x < s$.

Solution:

Suppose that no such s exists, for contradiction. Then for every $s \in S$, we have that $s \leq x$. Then x is an upper bound for S by definition. Since y is a least upper bound of S, then $y \leq x$. But $x < y$, which is a contradiction.

8. Does the empty set have a least upper bound in $\mathbb{R}$? If yes, find it and prove it is a least upper bound. If no, prove that no least upper bound exists.

Solution:

Let $x \in \mathbb{R}$. Then $x > s$ for all $s \in \varnothing$, since there are no elements in $\varnothing$. Thus x is an upper bound for $\varnothing$.

Suppose y is an upper bound for $\varnothing$. Then $y - 1 < y$, and $y - 1 \in \mathbb{R}$, so $y - 1$ is an upper bound for $\varnothing$, and thus y is not a least upper bound. So no least upper bound exists.

9. Let S be an ordered set and $A \subset S$. Suppose b is an upper bound for A, and that $b \in A$. Prove that $b = \sup A$.

   Solution:

   Since b is an upper bound for A, we know that $a \leq b$ for all $a \in A$.

   Suppose y is an upper bound for A. Then $a \leq y$ for all $a \in A$. Since $b \in A$, this means that $b \leq y$.

   Thus b is an upper bound, and $b \leq y$ for every upper bound y. By definition of least upper bound, b is the least upper bound of A.