Math 310 Fall 2018
Real Analysis HW 2 Solutions
Due Friday, September 14

You may *not* discuss the starred problem with classmates, though you should of course feel free to discuss it with me as much as you like. Linguistic precision is important for this problem. Submit this problem on a separate, detached sheet of paper.

$\star$ **Redo Problem:** Let S_1, S_2 be non-empty subsets of $\mathbb{R}$ that are bounded above. Let $S = \{s_1 + s_2 : s_1 \in S_1, s_2 \in S_2\}$. Prove that $\sup S = \sup S_1 + \sup S_2$.

For the remainder of these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. Prove that 1 is the least upper bound of $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$.

   Solution: First we claim that 1 is an upper bound for S. Since $1/n > 0$ we know that $-1/n < 0$ and thus $1 - 1/n < 1$, so 1 is an upper bound.

   Suppose $y < 1$. Then $1 - y > 0$, and by the corollary to the Archimedean property in number 5, there is a $n \in \mathbb{N}$ such that $1/n < 1 - y$. Then $-1/n > y - 1$ and so $1 - 1/n > y$. But $1 - 1/n \in S$, so y is not an upper bound for S.

   Thus if y is an upper bound for S, we know $y \geq 1$. So $1 = \sup(S)$ by definition.

2. Let S be a subset of an ordered field. We say that y is a *lower bound* for S if $y \leq S$ for all $s \in S$. We say that y is a *greatest lower bound* for S, and write $y = \inf(S)$ if y is a lower bound for S, and if x is also a lower bound for S then $y \geq x$.

   Prove that if S has a greatest lower bound then that lower bound is unique.

   Solution:

   Let x and y both be greatest lower bounds for S. Then x is a lower bound for S, so $y \geq x$ since y is a greatest lower bound.

   But y is a lower bound, so $x \geq y$ since x is a greatest lower bound.

   Since $y \geq x$ and $x \geq y$, we must have $y = x$.

3. Let $S \subset \mathbb{R}$ be non-empty and bounded below. Prove that S has a greatest lower bound.

   Solution: Let $T = \{-s : s \in S\}$. S is bounded below, so there is some x with $x \leq s$ for all $s \in S$. Then $-x \geq -s$ for all $s \in S$, so $-x \geq t$ for all $t \in T$. So T is bounded above, and thus by the least upper bound principle, T has a least upper bound.

Let $y = \sup(T)$. We claim that $-y = \inf(S)$. First we claim that $-y$ is a lower bound. If $s \in S$, then $-s \in T$ so $y \geq -s$. Thus $-y \leq s$. So y is a lower bound for S.

Now suppose z is also a lower bound for S. Then if $t \in T$, we know $-t \in S$ so $z \leq -t$ and thus $-z \geq t$. So $-z$ is an upper bound for T, and thus $-z \geq y$ since $y = \sup(T)$. Thus $z \leq -y$. Since $z \leq -y$ for every lower bound z, we see that $-y$ is a greatest upper bound for S.

Alternate answer:

Let T be the set of lower bounds of S. We know that T is non-empty because S is bounded below. If $s \in S$ then $s \geq t$ for every $t \in T$, so T is bounded above. By the Least Upper Bound principle, T has a least upper bound.

Let $t = \sup(T)$. We claim that $t = \inf(S)$. First we need to show that t is a lower bound for S. Let $s \in S$ and suppose $s < t$. Then by problem 7 from last week's homework, there is a $t_1 \in T$ such that $t_1 > s$; but t_1 is a lower bound for S so this is impossible. Thus $s \geq t$ for all $s \in S$, and so t is a lower bound for S.

Now suppose t_1 is a lower bound for S. Then $t_1 \in T$, so $t_1 \leq t$. Thus t is the greatest lower bound of S by definition.

4. Let $S = \{\frac{1}{n^2} : n \in \mathbb{N}\}$. Find the greatest lower bound for S, and prove it is the greatest lower bound.

   Solution: We claim that $0 = \inf(S)$.

   For every $n \in N$, we know that $n > 0$, so $1/n > 0$, so $1/n^2 > 0$. Thus $0 < s$ for every $s \in S$.

   Now suppose $x > 0$. By the corollary to the Archimedean property in number 5, we know there is an n with $1/n < x$, and since $1/n \leq 1$ we have $1/n^2 \leq 1/n$. Thus $1/n^2 < x$, so x is not a lower bound for S.

   Therefore, if x is a lower bound for S, then $x \leq 0$. Thus $0 = \inf(S)$.

5. Prove that for any real number $\epsilon > 0$, there is a $n \in \mathbb{N}$ such that $1/n < \epsilon$.

   Solution: Since $\epsilon > 0$, we also have $1/\epsilon > 0$. By the Archimedean property, there is an $n \in \mathbb{N}$ such that $n \geq 1/\epsilon > 0$. Then $1/n \leq \epsilon$.

6. Prove that if $x < y$, then $x^3 < y^3$.

   Solution:

   We divide this into three cases.

   Suppose $x, y \neq 0$. Then $x^2, xy, y^2 > 0$. Multiplying $x < y$ by x^2 gives us $x^3 < x^2y$; multiplying by xy gives $x^2y < xy^2$; and multiplying by y^2 gives $xy^2 < y^3$. Transitivity then gives $x^3 < y^3$.

   If $x = 0$ then we have $0 < y$, so $0^3 = 0 < y^3$ implies that $x^3 < y^3$.

   If $y = 0$ then we have $x < 0$. We still know $x^2 > 0$, and multiplying gives $x \cdot x^2 < 0 = 0^3$ so $x^3 < y^3$.

7. Prove that if x is a real number, there is a unique real number y such that $y^3 = x$.

   Solution:

If $y_1 < y_2$ then $y_1^3 < y_2^3$, so $y_1^3 \neq y_2^3$. Thus cube roots are unique. We just need to show they exist.

Let $S = \{a \in \mathbb{R} : a^3 < x\}$. Then S is non-empty, and S is bounded above by $\max\{1, x\}$ since $a^3 < a$ when $a \geq 1$. By the Least Upper Bound principle, we know that S has a least upper bound; let $y = \sup(S)$. We claim that $y^3 = x$.

For any real number $\epsilon > 0$ we know that

$$y - \epsilon < y < y + \epsilon$$
$$(y - \epsilon)^3 < y^3 < (y + \epsilon)^3.$$

But we also know that $(y - \epsilon)^3 < x$ and $(y + \epsilon)^3 > x$ because $y = \sup(S)$. Thus we have

$$(y - \epsilon)^3 < x < (y + \epsilon)^3$$
$$(y - \epsilon)^3 - (y + \epsilon)^3 < y^3 - x < (y + \epsilon)^3 - (y - \epsilon)^3$$
$$-6y^2\epsilon - 6y\epsilon^3 < y^3 - x < 6y^2\epsilon + 6y\epsilon^2.$$

Thus $|y^3 - x| < 6y\epsilon(y + \epsilon)$ for any $\epsilon > 0$.

Now let $a > 0$ be any positive real number; we can choose ϵ such that $6y\epsilon(y + \epsilon) < a$ (by, for instance, taking $\epsilon < y, \frac{1}{12y^2}$. Thus $|y^3 - x| < a$ for any positive real number a; so $|y - x| = 0$ and $y^3 = x$.

8. Let $a < b$ be real numbers. Prove there is a rational number r with $a < r < b$.

   Solution: $b - a > 0$ so there is a natural number N with $1/N < b - a$. Then we know there is a $n \in \mathbb{Z}$ such that $\frac{n}{N} \leq b < \frac{n+1}{N}$.

   Clearly $\frac{n}{N}$ is rational. We claim that $a < \frac{n}{N}$. Otherwise we have

$$\frac{n}{N} \leq a$$
$$-a \leq -\frac{n}{N}$$
$$b < \frac{n+1}{N}$$
$$b - a < \frac{1}{N}$$

   which is a contradiction.

   If $\frac{n}{N} < b$, then we're done, since we have $a < \frac{n}{N} < b$.

   Now suppose $\frac{n}{N} = b$. Then since $\frac{1}{N} < b - a$ we have $a < b - \frac{1}{N} < b$. But $b - \frac{1}{N} = \frac{n-1}{N}$ is a rational number, so we're done.

9. Prove that $\mathbb{R}^2$ with the metric $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$ is a metric space.

   Solution:

   We need to check three things.

(a) (Non-negativity) It's clear that $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2| \geq 0$. Suppose that $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2| = 0$. Then since $|x_i - y_1| \geq 0$, we must have $|x_1 - y_1| = |x_2 - y_2| = 0$ and thus $x_1 = y_1, x_2 = y_2$. Thus if $d(\vec{x}, \vec{y}) = 0$ then $\vec{x} = \vec{y}$.

(b) (Symmetry)

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2| = d((y_1, y_2), (x_1, x_2)).$$

(c) (Triangle inequality)

$$\begin{aligned} d((x_1, x_2), (z_1, z_2)) &= |x_1 - z_1| + |x_2 - z_2| = |x_1 - y_1 + y_1 - z_1| + |x_2 - y_2 + y_2 - z_2| \\ &\leq |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2| \\ &= d((x_1, x_2), (y_1, y_2)) + d((y_1, y_2), (x_1, x_2)). \end{aligned}$$