Math 310 Fall 2018
Real Analysis HW 3
Due Friday, September 21

You may *not* discuss the starred problem with classmates, though you should of course feel free to discuss it with me as much as you like. Linguistic precision is important for this problem. Submit this problem on a separate, detached sheet of paper.

$\star$ **Redo Problem:** Let $E = \mathcal{C}([0,1])$ be the set of continuous functions from $[0,1]$ to $\mathbb{R}$. The L^1 metric is given by $d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx$. Prove this is a metric.

For the remainder of these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. Let $E = \mathcal{B}([0,1], \mathbb{R})$ be the set of bounded functions from the closed interval $[0,1]$ to $\mathbb{R}$—that is, the set of all functions $f : [0,1] \to \mathbb{R}$ such that there is some number y with $|f(x)| \leq y$ for all $x \in [0,1]$. Define $d_{sup}(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$. Show that this is a metric.

2. Let $E = \mathbb{R}^2$ and define

$$d((x_1,y_1),(x_2,y_2)) = \begin{cases} |y_1| + |y_2| + |x_1 - x_2| & x_1 \neq x_2 \\ |y_1 - y_2| & x_1 = x_2 \end{cases}$$

   Prove that this is a metric on $\mathbb{R}^2$.

3. For each of the following metric spaces, describe the open ball of radius 1 centered at the given point.

   (a) $E = \mathcal{B}([0,1], \mathbb{R})$ with the sup metric, around the point $f(x) = 0$.

   (b) $E = \mathcal{C}([0,1], \mathbb{R})$ with the L^1 metric, around the point $f(x) = 0$. (The L^1 metric is defined in the redo problem).

   (c) $E = \mathbb{R}^2$ with the metric given in number (2), around the point $(0,0)$.

4. Let E be a set under the discrete metric, and let $S \subset E$. Prove S is open. Prove S is closed.

5. Let $E = \mathbb{R}^n$ under the Euclidean metric, and let $S = \bigcap_{n \in \mathbb{N}} B_{1/n}(\vec{0})$. Describe the set S geometrically. Is it open, closed, both, or neither? Why does this not contradict Proposition 2.15?

6. Show that the set $U = \{(x_1, x_2) : x_1 > x_2\}$ is open in $\mathbb{R}^2$ with one of the Euclidean, sup, or sum metric.

7. Show that the set $V = \{(x_1, x_2) : x_1 x_2 = 1, x_1 > 0\}$ is closed in $\mathbb{R}^2$ with one of the Euclidean, sup, or sum metric.

   (You don't need to turn this in, but think about why your proof doesn't work without the $x_1 > 0$ condition).

8. If E is a metric space, $x \in E$, and $r > 0$, prove that $\overline{B}_r(x)$ is a closed set.

9. Let E be a metric space, and suppose $x_1, x_2, \dots$ is a sequence that converges to x in E (that is, $\lim_{n\to\infty} x_n = x$). Prove that the sequence $x_1, x, x_2, x, x_3, \dots$ converges to x.

   (For notational simplicity, you can refer to the sequence $x_1, x, \dots$ as (y_m). So $y_1 = x_1$ and $y_2 = x$ and $y_3 = x_2$ and so on).