# Math 310 Fall 2018
# Real Analysis HW 3 Solutions
# Due Friday, September 21

You may *not* discuss the starred problem with classmates, though you should of course feel free to discuss it with me as much as you like. Linguistic precision is important for this problem. Submit this problem on a separate, detached sheet of paper.

$\star$ **Redo Problem:** Let $E = \mathcal{C}([0,1])$ be the set of continuous functions from $[0,1]$ to $\mathbb{R}$. The $L^1$ metric is given by $d_1(f,g) = \int_0^1 |f(x) - g(x)|\,dx$. Prove this is a metric.

For the remainder of these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. Let $E = \mathcal{B}([0,1],\mathbb{R})$ be the set of bounded functions from the closed interval $[0,1]$ to $\mathbb{R}$—that is, the set of all functions $f : [0,1] \to \mathbb{R}$ such that there is some number $y$ with $|f(x)| \leq y$ for all $x \in [0,1]$. Define $d_{sup}(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$. Show that this is a metric.

   **Solution:**

   (a) $d_{sup}(f,g) = \sup\{|f(x) - g(x)|\} \geq |f(0) - g(0)| \geq 0$ since $|x| \geq 0$ for all $x$.
   if $d_{sup}(f,g) = 0$, then $\sup\{|f(x) - g(x)|\} = 0$. This implies that $|f(x) - g(x)| = 0$ for every $x \in [0,1]$, and thus $f(x) = g(x)$ for every $x \in [0,1]$. Thus the two functions are the same.

   (b) We know $|f(x) - g(x)| = |g(x) - f(x)|$, so

   $$d_{sup}(f,g) = \sup\{|f(x) - g(x)|\} = \sup\{|g(x) - f(x)|\} = d_{sup}(g,f).$$

   (c) For each $x$, we have

   $$|f(x) - h(x)| = |f(x) - g(x) + g(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)|$$

   by the triangle inequality. Thus

   $$\begin{aligned} d_{sup}(f,h) &= \sup\{|f(x) - h(x)|\} = \sup\{|f(x) - g(x) + g(x) - h(x)|\} \\ &\leq \sup\{|f(x) - g(x)| + |g(x) - h(x)|\} \\ &\leq \sup\{|f(x) - g(x)|\} + \sup\{|g(x) - h(x)|\} = d_{sup}(f,g) + d_{sup}(g,h). \end{aligned}$$

2. Let $E = \mathbb{R}^2$ and define

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1| + |y_2| + |x_1 - x_2| & x_1 \neq x_2 \\ |y_1 - y_2| & x_1 = x_2 \end{cases}$$

Prove that this is a metric on $\mathbb{R}^2$.

**Solution:**

(a) The output is always a sum of absolute values, and thus $\geq 0$. If $d((x_1, y_1), (x_2, y_2)) = 0$, then it must be the case that $x_1 = x_2$; otherwise, we'd have $|x_1 - x_2| > 0$ and thus $|y_1| + |y_2| + |x_1 - x_2| > 0$. But if $x_1 = x_2$, then $0 = |y_1 - y_2|$ which implies that $y_1 = y_2$, and thus $(x_1, y_1) = (x_2, y_2)$.

(b)

$$\begin{aligned} d((x_1, y_1), (x_2, y_2)) &= \begin{cases} |y_1| + |y_2| + |x_1 - x_2| & x_1 \neq x_2 \\ |y_1 - y_2| & x_1 = x_2 \end{cases} \\ &= \begin{cases} |y_2| + |y_1| + |x_2 - x_1| & x_1 \neq x_2 \\ |y_2 - y_1| & x_1 = x_2 \end{cases} \\ &= d((x_2, y_2), (x_1, y_1)). \end{aligned}$$

(c) We need to prove that $d((x_1, y_1), (x_3, y_3)) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3))$. We divide this into cases.

**Case 1:** $x_1 = x_2 = x_3$

$$\begin{aligned} d((x_1, y_1), (x_3, y_3)) = |y_1 - y_3| &\leq |y_1 - y_2| + |y_2 - y_3| \\ &= d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), d(x_3, y_3)). \end{aligned}$$

**Case 2:** $x_1 = x_2 \neq x_3$

$$\begin{aligned} d((x_1, y_1), (x_3, y_3)) &= |y_1| + |y_3| + |x_1 - x_3| = |y_1 - y_2 + y_2| + |y_3| + |x_2 - x_3| \\ &\leq |y_1 - y_2| + (|y_2| + |y_3| + |x_2 - x_3|) \\ &= d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)). \end{aligned}$$

**Case 3:** $x_1 \neq x_2 = x_3$ is the same as Case 2.

**Case 4:** $x_1 = x_3 \neq x_2$

$$\begin{aligned} d((x_1, y_1), (x_3, y_3)) &= |y_1 - y_3| \leq |y_1| + |-y_3| \\ &\leq |y_1| + |y_2| + |x_1 - x_2| + |y_2| + |y_3| + |x_2 - x_3| \\ &= d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), d(x_3, y_3)). \end{aligned}$$

**Case 5:** $x_1, x_2, x_3$ all distinct.

$$\begin{aligned} d((x_1, y_1), (x_3, y_3)) &= |y_1| + |y_3| + |x_1 - x_3| = |y_1| + |y_3| + |x_1 - x_2 + x_2 - x_3| \\ &\leq |y_1| + |y_3| + |x_1 - x_2| + |x_2 - x_3| \\ &\leq |y_1| + |y_2| + |x_1 - x_2| + |y_2| + |y_3| + |x_2 - x_3| \\ &= d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), d(x_3, y_3)). \end{aligned}$$

3. For each of the following metric spaces, describe the open ball of radius 1 centered at the given point.

   (a) $E = \mathcal{B}([0,1], \mathbb{R})$ with the sup metric, around the point $f(x) = 0$.

   (b) $E = \mathcal{C}([0,1], \mathbb{R})$ with the $L^1$ metric, around the point $f(x) = 0$. (The $L^1$ metric is defined in the redo problem).

   (c) $E = \mathbb{R}^2$ with the metric given in number (2), around the point $(0,0)$.

   **Solution:**

   (a) The set of all functions $f : [0,1] \to \mathbb{R}$ such that $-1 < f(x) < 1$ for every $x \in [0,1]$.

   (b) The set of all continuous functions $f : [0,1] \to \mathbb{R}$ such that $\int_0^1 |f(x)|\,dx < 1$.

   (c) This turns out to be the same as the unit open ball in the sum metric: it's a diamond with corners at $(0,1), (1,0), (-1,0), (0,-1)$. It doesn't contain the sides but it is the entire interior.

4. Let $E$ be a set under the discrete metric, and let $S \subset E$. Prove $S$ is open. Prove $S$ is closed.

   **Solution:**

   If $S = \varnothing$ then we know $S$ is open. So suppose $S$ is non-empty and let $x \in S$. Then $B_1(x) = \{x\}$ under the discrete metric, since $d(x,y) = 1$ for all $y \neq x$. We see that $B_1(x) = \{x\} \subset S$ since $x \in S$. Thus by definition, $S$ is open.

   Now consider $S^C$. By the previous argument, we know that $S^C$ is open. Thus $S$ is closed.

5. Let $E = \mathbb{R}^n$ under the Euclidean metric, and let $S = \bigcap_{n\in\mathbb{N}} B_{1/n}(\vec{0})$. Describe the set $S$ geometrically. Is it open, closed, both, or neither? Why does this not contradict Proposition 2.15?

   **Solution:** Clearly $0 \in B_{1/n}(\vec{0})$ for every $n \in \mathbb{N}$. We claim that $\{\vec{0}\} = B_{1/n}(\vec{0})$.

   Suppose $\vec{x} \in \mathbb{R}^n$ is distinct from the origin. Then set $r = d(\vec{x}, \vec{0}) > 0$. By a corollary to the Archemedean property, we can find some $n$ such that $1/n < r$.

   Then $d(\vec{x}, \vec{0}) > 1/n$, so $\vec{x} \notin B_{1/n}(\vec{0})$ by definition. Thus $\vec{x} \notin S = \bigcap_{n\in\mathbb{N}} B_{1/n}(\vec{0})$.

   Thus $S = \{\vec{0}\}$ since no other point can be an element. This single-point set is closed, but is not open since no open ball contains only one point.

   This doesn't contradict proposition 2.15 because the proposition says any finite intersection of open sets is open. But this is an infinite intersection of open sets, so doesn't need to be open.

6. Show that the set $U = \{(x_1, x_2) : x_1 > x_2\}$ is open in $\mathbb{R}^2$ with one of the Euclidean, sup, or sum metric.

   **Solution:**

   **Sup Metric:** Let $(x_1, x_2) \in U$. Then we know $x_1 > x_2$. Let $r = x_1 - x_2 > 0$. We claim $B_{r/2}(x_1, x_2) \subset U$ under the sup metric.

Let $(x, y) \in B_{r/2}(x_1, x_2)$. Then $d((x, y), (x_1, x_2)) < r/2$, so $\sup\{|x-x_1|, |y-x_2|\} < r/2$. This implies that $|x - x_1| < r/2$ and $|y - x_2| < r/2$. So we know that $y < x_2 + r/2$ and $x_1 < x + r/2$.

But recall that $x_1 = x_2 + r$. Thus we have $x_2 + r < x + r/2$, and thus $x_2 + r/2 < x$. Transitivity gives us that $y < x$, and thus $(x, y) \in U$.

**Sum Metric:** Let $(x_1, x_2) \in U$. Then we know $x_1 > x_2$. Let $r = x_1 - x_2 > 0$. We claim $B_r(x_1, x_2) \subset U$ under the sum metric.

Let $(x, y) \in B_r(x_1, x_2)$. Then $d((x, y), (x_1, x_2)) < r$, so $|x - x_1| + |y - x_2| < r$. By the triangle inequality $|x - x_1 - y + x_2| \leq |x - x_1| + |y - x_2| < r$. Then we have

$$-r < x - y + x_2 - x_1 < r x_1 - x_2 - r \qquad < x - y < r + x_1 - x_2$$
$$0 < x - y < 2r$$

since $x_1 - x_2 = r$. Thus $0 < x - y$, so $x > y$, so $(x, y) \in U$. Thus $U$ is open by definition.

**Euclidean Metric:** You're going to have to take a ball of radius $r/\sqrt{2}$ or something like that. It's way messier than either of the others, for no really good reason.

7. Show that the set $V = \{(x_1, x_2) : x_1 x_2 = 1, x_1 > 0\}$ is closed in $\mathbb{R}^2$ with one of the Euclidean, sup, or sum metric.

   (You don't need to turn this in, but think about why your proof doesn't work without the $x_1 > 0$ condition).

   **Solution:**

   To be posted next week.

8. If $E$ is a metric space, $x \in E$, and $r > 0$, prove that $\overline{B}_r(x)$ is a closed set.

   **Solution:**

   Suppose $y \notin \overline{B}_r(x)$. Then $d(y, x) > r$. Let $s = d(y, x) - r$; we claim that $B_s(y) \subset \overline{B}_r(x)^C$.

   Suppose $z \in B_s(y)$. Then $d(z, y) < s$. By the triangle inequality,

   $$d(z, x) \geq d(x, y) - d(y, z) > d(x, y) - s = d(x, y) - (d(x, y) - r) = r.$$

   Thus $d(z, x) > r$, so $d(z, x) \in \overline{B}_r(x)^C$.

   This proves that $\overline{B}_r(x)^C$ is open, and thus the closed ball is closed, as desired.

9. Let $E$ be a metric space, and suppose $x_1, x_2, \ldots$ is a sequence that converges to $x$ in $E$ (that is, $\lim_{n\to\infty} x_n = x$). Prove that the sequence $x_1, x, x_2, x, x_3, \ldots$ converges to $x$.

   (For notational simplicity, you can refer to the sequence $x_1, x, \ldots$ as $(y_m)$. So $y_1 = x_1$ and $y_2 = x$ and $y_3 = x_2$ and so on).

   **Solution:**

   Let $\epsilon > 0$. Then there is some $N$ such that if $n > N$ then $d(x_n, x) < \epsilon$.

   Let $M = 2N$. Then if $m < M$, we have two possibilities. If $m$ is even then $y_m = x$, and $d(y_m, x) = d(x, x) = 0 < \epsilon$. If $m$ is odd, then $y_m = x_{(m+1)/2}$, and we have $(m + 1)/2 > M/2 = N$ so $d(y_m, x) = d(x_{(m+1)/2}, x) < \epsilon$.

   Thus either way, $d(y_m, x) < \epsilon$ if $m > M$, so by definition, $\lim_{m\to\infty} y_m = x$.