Math 310 Fall 2018
Real Analysis HW 4 Solutions
Due Friday, September 28

You may *not* discuss the starred problem with classmates, though you should of course feel free to discuss it with me as much as you like. Linguistic precision is important for this problem. Submit this problem on a separate, detached sheet of paper.

$\star$ **Redo Problem:** Let E be a metric space under the discrete metric. Prove that $\lim_{n\to\infty} x_n = x$ converges if and only if there is some $N \in \mathbb{N}$ such that $x_n = x$ for every $n > N$.

For the remainder of these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. If you didn't already, finish problem 7 from last homework: show that the set $V = \{(x_1, x_2) : x_1x_2 = 1, x_1 > 0\}$ is closed in $\mathbb{R}^2$ with one of the Euclidean, sup, or sum metric.

   Solution:

   Fix (x_1, x_2) such that $x_1x_2 \neq 1$ and $x_1 > 0$. We want to find some open ball $B_\epsilon(x_1, x_2)$ so that if $(y_1, y_2) \in B_\epsilon(x_1, x_2)$, then $y_1y_2 \neq 1$.

   Let $\epsilon = |x_1x_2 - 1| > 0$. If $|x_1x_2 - y_1y_2| < \epsilon$, we have

$$
\begin{aligned}
|x_1x_2 - y_1y_2| &< |x_1x_2 - 1| \\
0 < |x_1x_2 - 1| - |x_1x_2 - y_1y_2| &\leq |x_1x_2 - 1 - x_1x_2 + y_1y_2| \\
&= |y_1y_2 - 1|
\end{aligned}
$$

   and thus $y_1y_2 \neq 1$, so $y_1y_2 \in U$. So we just need to find some open ball so that if (y_1, y_2) is in the ball, then $|x_1x_2 - y_1y_2| < \epsilon$.

   If $(y_1, y_2) \in B_\delta(x_1, x_2)$, then we see that $|x_1 - y_1| < \delta$ and $|x_2 - y_2| < \delta$ in whichever metric we choose. Then

$$
\begin{aligned}
|x_1x_2 - y_1y_2| &= |x_1x_2 - x_1y_2 + x_1y_2 - y_1y_2| \\
&\leq |x_1x_2 - x_1y_2| + |x_1y_2 - y_1y_2| \\
&= |x_1| \cdot |x_2 - y_2| + |y_2| \cdot |x_1 - y_1| \\
&< |x_1|\delta + |y_2|\delta \\
&= \delta|x_1| + \delta|y_2 - x_2 + x_2| \leq \delta|x_1| + \delta|y_2 - x_2| + \delta|x_2| \\
&< \delta(|x_1| + |x_2| + \delta).
\end{aligned}
$$

So choose δ such that $\delta(|x_1| + |x_2| + \delta) < \epsilon$. Then if $(y_1, y_2) \in B_\delta(x_1, x_2)$, we have $|x_1x_2 - y_1y_2| < \epsilon = |x_1x_2 - 1|$, and thus $|y_1y_2 - 1| > 0$ and so $(y_1, y_2) \in U$.

Optional:

We can show that such a δ exists like this. If $\delta < 1$, then $\delta(|x_1| + |x_2| + \delta) < \delta(|x_1| + |x_2| + 1)$. Then if $\delta < \frac{\epsilon}{|x_1|+|x_2|+1}$ then we have $\delta(|x_1| + |x_2| + 1) < \epsilon$. So we just need to pick some δ that is less than $\min\left\{1, \frac{\epsilon}{|x_1|+|x_2|+1}\right\}$.

2. Let (E, d) be a metric space, and let $S \subset E$. Prove that S is bounded if and only if the set $\{d(x, y) : x, y \in S\} \subset \mathbb{R}$ is bounded above.

Solution: Suppose S is bounded. Then there exists some $x \in E$ and some $r > 0$ such that $S \subset B_r(x)$. Let $y, z \in S$. Then $d(y, z) \leq d(y, x) + d(x, z)$ by the triangle inequality, and $d(y, x), d(x, z) < r$ since $y, z \in B_r(x)$. Thus $d(y, z) < 2r$. This is true for any $y, z \in S$, so $2r$ is an upper bound for $\{d(x, y) : x, y \in S\}$.

Now suppose D is an upper bound for $\{d(x, y) : x, y \in S\}$. Fix some $x \in S$. Then for any $y \in S$, $d(x, y) < D$ since D is an upper bound for the set of distances, so $y \in B_D(x)$. Thus $S \subset B_D(x)$, and S is bounded by definition.

3. Let $E = \mathbb{R}^2$ and consider the sequence $\left(\frac{1}{n}, 1 - \frac{1}{n}\right) = (1, 0), (1/2, 1/2), (1/3, 2/3), \ldots$. Prove that this sequence converges in one of the sup, sum, or Euclidean metric.

Solution:

Sup metric: Let $\epsilon > 0$, and let $N > 1/\epsilon$. If $n > N$ then

$$d(x_n, (0, 1)) = \max\{|1/n - 0|, |(1 - 1/n) - 1|\} = \max\{|1/n|, |1/n|\} = 1/n < 1/N < \epsilon.$$

Thus by definition, $\lim_{n\to\infty} x_n = (0, 1)$.

4. Let $E = \mathbb{R}$ and let $x_n = n$. Prove that (x_n) does not converge in the regular absolute value metric.

Solution:

Suppose $\lim_{n\to\infty} x_n = L$ for some $L \in \mathbb{R}$. Then for every $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|x_n - L| < \epsilon$ for all $n > N$. Then $|N + 1 - L| < \epsilon$ and $|N + 2 - L| < \epsilon$; by the triangle inequality, we get $2\epsilon > |N + 2 - L + L - N - 1| = |1|$. Thus for all $\epsilon > 0$ we have $1 < 2\epsilon$, which is a contradiction.

Alternate proof:

Suppose $\lim_{n\to\infty} x_n = L$ for some $L \in \mathbb{R}$. Then for every $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|x_n - L| < \epsilon$ for all $n > N$. We can choose a n such that $n > N$ and also $n > L + 1$ by the Archimedean property; then we have $\epsilon > |n - L| = n - L > 1$. Thus for all $\epsilon > 0$ we have $\epsilon > 1$, which is a contradiction.

5. Let (E, d) be a metric space, and let V be a closed subset of E. Prove that $\overline{V} = V$.

Solution:

We know that $V \subset \overline{V}$. (This was stated in class, but also $\overline{V}$ is an intersection of sets all of which contain V).

Conversely, $\overline{V}$ is the intersection of all closed sets that contain V. But V is itself a closed set that contains V, so $\overline{V}$ is an intersection of V with some other sets; so $\overline{V} \subset V$.

6. Let (E, d) be a metric space, and $x \in E$. Is the closure of $B_r(x)$ always equal to the closed ball $\overline{B}_r(x)$? Either prove it, or find a counterexample.

   Solution:

   Let E be a set under the discrete metric, and let $x \in E$. Then $B_1(x) = \{x\}$ is a closed set, so $\overline{B_1(x)} = \{x\}$. But the closed ball $\overline{B}_1(x) = E$ since every point has distance 1 from x. Thus the closed ball is not the closure of the open ball.

7. If (E, d) is a metric space and $U \subset E$, prove the interior of U is the set of all points $x \in U$ such that some open ball containing x is also a subset of U.

   Solution:

   We want to prove that two sets are equal, so we prove that each is a subset of the other.

   Let $x \in \mathring{U}$. We know $\mathring{U} = \bigcup_{V \subset U V \text{open}} V$ is the union of all open sets inside U. So there is some open set V with $x \in V \subset U$. Then by definition of open set, there is an open ball $B_r(x) \subset V$, and thus $B_r(x) \subset U$. So every point in $\mathring{U}$ has some open ball containing it that's a subset of U.

   Conversely, suppose $x \in U$ and some open ball B containing x is a subset of U. Then B is an open subset of U, so $B \subset \mathring{U}$. Thus $x \in \mathring{U}$.

8. Consider the metric space $\mathbb{R}$ under the usual metric. Find the interior of $[0, 1]$ and prove your answer.

   Solution:

   We claim the interior $\mathring{U}$ is $(0, 1)$.

   First we observe that $(0, 1)$ is an open subset of $[0, 1]$. Thus $(0, 1)$ must be a subset of $\mathring{U}$.

   Now we claim that $0, 1 \notin \mathring{U}$. For any $r > 0$, we have $1+r/2 \in B_r(1)$ but $1+r/2 \notin [0, 1]$, so $1 \not{i}n\mathring{U}$. Similarly, $-r/2 \in B_r(0)$ but $-r/2 \not{i}n[0, 1]$, so $0 \notin \mathring{U}$. Thus $\mathring{U}$ is precisely $(0, 1)$.