Math 310 Fall 2018
Real Analysis HW 5 Solutions
Due Wednesday, October 3

No redo problem this week.

For the remainder of these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, and $a_n \leq b_n$ for all n, prove that $a \leq b$.

   Solution:

   Suppose $b < a$, and let $\epsilon = a - b$. Then there exists some N_1 so that $|a_n - a| < \epsilon/2$ for $n > N_1$, and some N_2 so that $|b_n - b| < \epsilon/2$ for $n > N_2$.

   Let $N = \max\{N_1, N_2\}$. Then if $n > N$, we have

$$\begin{aligned}
|b_n - b| &< \epsilon/2 \\
b_n - b &< \epsilon < 2 \\
b_n &< b + \epsilon/2 \\
|a_n - a| &< \epsilon/2 \\
a_n - a &> -\epsilon/2 \\
a_n &> a - \epsilon/2 = b + \epsilon/2
\end{aligned}$$

   and thus we have $b_n < b + \epsilon/2 < a_n$ by transitivity; but this is a contradiction.

2. Let $a_n, b_n \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Prove that $\lim_{n\to\infty}(a_n b_n) = ab$.

   Solution: Let $\epsilon > 0$. Then there is a N_1 so that $|a_n - a| < \frac{\epsilon}{2|b|+2}$ when $n < N_1$, adn there is a N_2 so that $|b_n - b| < \min\{1, \epsilon/2|a|\}$ when $n > N_2$.

   Let $N = \max\{N_1, N_2\}$, and suppose $n > N$. Then

$$\begin{aligned}
|a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\
&\leq |a_n b_n - ab_n| + |ab_n - ab| \\
&= |a_n - a| \cdot |b_n| + |a||b_n - b|.
\end{aligned}$$

   Now, we want to show that $|b_n|$ can't be too much bigger than b. So we see

$$|b_n| = |b_n - b + b| \leq |b_n - b| + |b| < |b + 1|$$

and thus we have

$$\begin{aligned}|a_nb_n - ab| &< |a_n - a|(|b|+1) + |a||b_n - b| \\ &< \frac{\epsilon}{2|b|+b}(|b|+1) + |a|\frac{\epsilon}{2|a|} \\ &= \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

(We technically should worry about what happens when $a = 0$, in which case you have to modify the proof a bit. But I don't want to take off any points for missing that).

3. Let $S \subset \mathbb{R}$ be nonempty and bounded above. Prove there is a monotone sequence (x_n) such that $x_n \in S$ and $\lim_{n\to\infty} x_n = \sup S$.

   Solution:

   For each $n \in \mathbb{N}$, there is some $x_n \in S$ so that $\sup S - \frac{1}{n} < x_n < \sup S$ (by HW1 problem 7). Thus $|x_n - \sup S| = \sup S - x_n < \frac{1}{n}$.

   Now we claim $\lim_{n\to\infty} x_n = \sup S$. Let $\epsilon > 0$, and let $N > \frac{1}{\epsilon}$. Then if $n > N$, we have

$$|x_n - \sup S| < \frac{1}{n} < \frac{1}{N} < \epsilon.$$