Math 310 Fall 2018
Real Analysis HW 6
Due Friday, October 19

You may *not* discuss the starred problem with classmates, though you should of course feel free to discuss it with me as much as you like. Linguistic precision is important for this problem. Submit this problem on a separate, detached sheet of paper.

$\star$ **Redo Problem:** Let (x_n) be a Cauchy sequence. Prove that any subsequence (x_{n_k}) of (x_n) is Cauchy.

For the remainder of these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. Let (a_n) and (b_n) be bounded sequences of real numbers. Prove that

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n.$$

2. (a) Find a pair of sequences of real numbers $(a_n), (b_n)$ so that

$$\limsup(a_n + b_n) < \limsup a_n + \limsup b_n.$$

   (b) Find a pair of sequences of real numbers $(a_n), (b_n)$ so that

$$\liminf(a_n + b_n) > \liminf a_n + \liminf b_n.$$

3. Let $x_n = \dfrac{(n-1)(-1)^n}{n}$. Find (with proof) $\limsup x_n$ and $\liminf x_n$.

4. Prove the Squeeze Theorem: Let $(a_n), (b_n), (x_n)$ be sequences of real numbers such that $a_n \leq x_n \leq b_n$ for all $n \in \mathbb{N}$. Suppose (a_n) and (b_n) converge, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Then (x_n) converges, and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} a_n$.

   (Hint: you do have to prove that x_n converges, and not just that if it converges, this is the limit).

5. Prove that any Cauchy sequence is bounded.

6. If E is a metric space under the discrete metric, prove it is complete.

7. (a) Give an example of an open set in some metric space that is complete. (You should prove that it is open and complete, but you can use results we've proven in class).

(b) Give an example of a closed set in some metric space that is not complete. (You should prove that it is closed and not complete, but you can use results we've proven in class).

8. Let (E, d) be a metric space, and supposed $V \subset E$ is complete. Prove that V is closed.