Math 310 Fall 2018
Real Analysis HW 6 Solutions
Due Friday, October 19

You may *not* discuss the starred problem with classmates, though you should of course feel free to discuss it with me as much as you like. Linguistic precision is important for this problem. Submit this problem on a separate, detached sheet of paper.

$\star$ **Redo Problem:** Let (x_n) be a Cauchy sequence. Prove that any subsequence (x_{n_k}) of (x_n) is Cauchy.

For the remainder of these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. Let (a_n) and (b_n) be bounded sequences of real numbers. Prove that

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n.$$

Solution:

For each n, if $k \geq n$, we know that $a_k + b_k \leq \sup\{a_k : k \geq n\} + \sup\{b_k : k \geq n\}$. Thus $\sup\{a_k + b_k : k \geq n\} \leq \sup\{a_k : k \geq n\} + \sup\{b_k : k \geq n\}$.

But the limit of the left-hand side as n goes to infinity is $\limsup(a_n + b_n)$, and the limit of the right-hand side is $\limsup a_n + \limsup b_n$.

2. (a) Find a pair of sequences of real numbers $(a_n), (b_n)$ so that

$$\limsup(a_n + b_n) < \limsup a_n + \limsup b_n.$$

Solution:

There are lots of examples; all of them will involve some sort of interference or cancellation. One good example is to take $(a_n) = 0, 1, 0, 1, \ldots$ and $(b_n) = 1, 0, 1, 0, \ldots$, so that $a_n + b_n = 1$ for all n. Then $\limsup a_n = 1$ and $\limsup b_n = 1$, but $\limsup(a_n + b_n) = 1 < 2 = \limsup a_n + \limsup b_n$.

(b) Find a pair of sequences of real numbers $(a_n), (b_n)$ so that

$$\liminf(a_n + b_n) > \liminf a_n + \liminf b_n.$$

Solution:

Again there are many examples, but if $(a_n) = 1, 2, 1, 2, \ldots$ and $(b_n) = 2, 1, 2, 1, \ldots$ then $\liminf a_n = \liminf b_n = 1$. But $a_n + b_n = 3$ for all n so $\liminf a_n + b_n = 3 > 2 = \liminf a_n + \liminf b_n$.

3. Let $x_n = \dfrac{(n-1)(-1)^n}{n}$. Find (with proof) $\limsup x_n$ and $\liminf x_n$.

Solution:

The sequence is $(x_n) = 0, 1/2, -2/3, 3/4, -4/5, 5/6, \ldots$. It's clear that $-1 \leq x_n \leq 1$ for all n; but if $r < 1$ then for all $n > 0$ there is a $k > n$ so that $x_k = 1 - 1/k > r$. Thus $\sup\{x_k : k > n\} = 1$ for each n, and thus $\limsup x_n = \lim_{n\to\infty} 1 = 1$.

Similarly, if $r > -1$ then for all $n > 0$ there is a $k > n$ so that $x_k = -1 + 1/k < r$. Thus $\inf\{x_k : k > n\} = -1$ for each n, and thus $\liminf x_n = \lim_{n\to\infty} -1 = 1$.

4. Prove the Squeeze Theorem: Let $(a_n), (b_n), (x_n)$ be sequences of real numbers such that $a_n \leq x_n \leq b_n$ for all $n \in \mathbb{N}$. Suppose (a_n) and (b_n) converge, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Then (x_n) converges, and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} a_n$.

(Hint: you do have to prove that x_n converges, and not just that if it converges, this is the limit).

Solution:

Let $\epsilon > 0$, and set $x = \lim_{n\to\infty a_n}$. Then there exist $N_1, N_2 \in \mathbb{N}$ so that if $n > N_1$ then $|a_n - x| < \epsilon/3$, and if $n > N_2$ then $|b_n - x| < \epsilon/3$.

Let $N = \max\{N_1, N_2\}$. Then if $n > N$, we know that $|a_n - x| < \epsilon/3$ and $|b_n - x| < \epsilon/3$. By the triangle inequality, we have $|a_n - b_n| \leq |a_n - x| + |x - b_n| < 2\epsilon/3$; and since $a_n \geq b_n$ this just tells us that $a_n - b_n < 2\epsilon/3$.

Since $x_n \geq b_n$ this then tells us that $a_n - x_n < 2\epsilon/3$, and since $a_n \geq x_n$ we know that $|a_n - x_n| < 2\epsilon/3$. Then $|x - x_n| \leq |x - a_n| + |a_n - x_n| < 2\epsilon/3 + \epsilon/2 = \epsilon$.

5. Prove that any Cauchy sequence is bounded.

Solution: Let (x_n) be a Cauchy sequence. Then there is some N so that if $m, n > N$ then $d(x_m, x_n) < 1$.

Let $R = \max\{d(x_{N+1}, x_1), d(x_{n+1}, x_2), \ldots, d(x_{N+1}, x_n)\}$. We claim that $\{x_n\} \subset B_{R+1}(x_{N+1})$.

If $n \leq N$ then $d(x_n, x_{N+1}) \leq R < R + 1$ so $x_n \in B_{R+1}(x_{N+1})$. If $n > N$ then $d(x_n, x_{N+1} < 1 \leq R + 1$, so $x_n \in B_{R+1}(x_{N+1})$. Thus (x_n) is bounded.

6. If E is a metric space under the discrete metric, prove it is complete.

Solution:

Let (x_n) be a Cauchy sequence. Then there is some N so that for all $m, n > N$, we have $d(x_m, x_n) < 1$. But then this means that for all $m, n > N$ we have $d(x_m, x_n) = 0$.

Let $x = x_{n+1}$, and let $\epsilon > 0$. Then for all $n > N$, we have $d(x_n, x) = 0 < \epsilon$.

7. (a) Give an example of an open set in some metric space that is complete. (You should prove that it is open and complete, but you can use results we've proven in class).

Solution: A few examples work, but they all have to cheat somehow; the standard open ball in $\mathbb{R}^n$ is definitely not complete. We can say that $\mathbb{R}$ is an open set and complete. Or we can say that $\varnothing$ is open, and is complete because, since it contains no Cauchy sequences, every Cauchy sequence it contains converges.

Or we can say that any set in the discrete metric is open (by a previous homework) and is complete (by the previous problem).

(b) Give an example of a closed set in some metric space that is not complete. (You should prove that it is closed and not complete, but you can use results we've proven in class).

Solution: As before, there are lots of choices, but they all have to cheat—and the metric space itself has to be incomplete.

The best answer is probably to let your metric space be the rational numbers, and then say that $\mathbb{Q}$ is closed and incomplete. But we could take any other infinite closed subset of $\mathbb{Q}$ and it would work as well.

8. Let (E, d) be a metric space, and supposed $V \subset E$ is complete. Prove that V is closed.

Solution: Let $V \subset E$ be a complete subset. Let (x_n) be a sequence in V that converges to some point $x \in E$. Then (x_n) converges and so is Cauchy; and since (x_n) is a Cauchy sequence in the complete space V, it has a limit in V. Thus $x \in V$.

Thus every sequence in V that converges in E has its limit in V; so V is a closed set.