Math 310 Fall 2018
Real Analysis HW 7 Solutions
Due Friday, October 26

You may *not* discuss the starred problem with classmates, though you should of course feel free to discuss it with me as much as you like. Linguistic precision is important for this problem. Submit this problem on a separate, detached sheet of paper.

$\star$ **Redo Problem:** Let (E, d) be a metric space, and suppose $\lim_{n\to\infty} x_n = x$. Prove that x is an accumulation point of the sequence (x_n).

For the remainder of these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. Let $S = \{\frac{1}{n} + \frac{1}{m} : n, m \in \mathbb{N}\}$. Find (with proof) all cluster points of S.

   Solution: We claim that the set of cluster points of S is $\{1/n : n \in \mathbb{N}\} \cup \{0\}$.

   Let $n \in \mathbb{N}$, and let $\epsilon > 0$. Then there is a m such that $1/m < \epsilon$, and so we have $d(1/n + 1/m, 1/n) = 1/m < \epsilon$. Thus $1/n + 1/m \in B_\epsilon(1/n)$, but $1/n + 1/m \neq 1/n$ and $1/n + 1/m \in S$. So $B_\epsilon(1/n)$ contains a point of $S \setminus \{1/n\}$ for every ϵ, so $1/n$ is an accumulation point of S.

   Now we wish to show that 0 is an accumulation point of S. Let $\epsilon > 0$. Then there is some number n so that $1/n < \epsilon/2$ so $2/n = 1/n + 1/n < \epsilon$. But $2/n \in S$ and $2/n \neq 0$. So $2/n \in B_\epsilon(0)$ and $2/n \in S \setminus \{0\}$. We can find such a point for any ϵ, and so 0 is an accumulation point of S.

2. Let (E, d) be a metric space, and let $V \subset E$. Prove that V is closed if and only if it contains all its accumulation points.

   Solution: First, suppose V is closed. Let x be an accumulation point of V. Then x is the limit of a sequence of points in $V \setminus \{x\}$; but since V is closed, this limit must be in V. So $x \in V$.

   Conversely, suppose V contains all of its accumulation points. We want to prove that if (x_n) is a sequence in V that converges, then $x = \lim_{n\to\infty} x_n$ is in V.

   We break this into two cases. If $x = x_n$ for some n, then clearly $x \in V$, since $x_n \in V$ for all n. So suppose $x_n \neq x$ for any n. Then for each $\epsilon > 0$, there is a $N \in \mathbb{N}$ so that if $n > N$, then $x_n \in B_\epsilon(x)$. Thus for each ϵ, there is a point of $V \setminus \{x\}$ in $B_\epsilon(x)$, and thus x is an accumulation point of V. But V contains all its accumulation points, so $x \in V$.

3. Let $S \subset \mathbb{R}$ be a set that is non-empty and bounded above, but has no greatest element. Prove $\sup S$ is a cluster point for S.

   Solution: Let $\epsilon > 0$. By HW1 problem 7, there is an $s \in S$ so that $s > \sup S - \epsilon$, and thus $s \in B_\epsilon(\sup S)$. But $s \neq \sup S$, since otherwise $s \geq t$ for any $t \in S$ and thus s is the greatest element of S. So $B_\epsilon(\sup S)$ contains an element of $S \setminus \{\sup(S)\} = S$ for each ϵ, and thus $\sup S$ is a cluster point of S.

   Remark: This claim is in fact false if S has a greatest element. If $S = \{1, 2\}$, then $\sup S = 2$ but 2 is not a cluster point of S.

4. Let $(x_n) = 0, 1, 2, 0, 1, 2, \ldots$. Prove that $0, 1, 2$ are all accumulation points of (x_n).

   Solution:

   Simple proof: The sequence (x_n) has subsequences $0, 0, 0, \ldots$, $1, 1, 1, \ldots$, and $2, 2, 2, \ldots$. The limits of these subsequences are $0, 1,$ and 2 respectively. Since any limit of a subsequence is an accumulation point for the sequence, this shows that $0, 1, 2$ are all accumulation points for x_n.

   More explicit proof: If $n \equiv 0 \mod 3$ then $x_n = 0 \in B_\epsilon(0)$ for every $\epsilon > 0$. Thus $B_\epsilon(0)$ contains x_n for infinitely many n, and so 0 is an accumulation point by definition.

   Similarly, if $n \equiv 1 \mod 3$ then $x_n = 1 \in B_\epsilon(1)$, so $B_\epsilon(1)$ contains x_n for infinitely many n, and 1 is an accumulation point. And if $n \equiv 2 \mod 3$ then $x_n = 2 \in B_\epsilon(2)$, so $B_\epsilon(2)$ contains x_n for infinitely many n, so 2 is an accumulation point.

5. Let $x_n = \begin{cases} 3/n & n \text{ odd} \\ 2 + 2/n & n \text{ even} \end{cases}$

   Find all accumulation points of x_n and prove they are accumulation points.

   Solution:

   Simple proof: We see that $\left(\frac{3}{2n-1}\right)$ is a subsequence of (x_n). Since it's also a subsequence of $3/n$ we know that it converges to 0, and thus 0 is an accumulation point of (x_n). Similarly, $(2 + 1/n)$ is a subsequence of (x_n), and it converges to 2, so 2 is an accumulation point of (x_n).

   More explicit proof: We want to show that 0 is an accumulation point for (x_n). For any $\epsilon > 0$ there exists an N such that $1/N < \epsilon/3$ and thus $3/N < \epsilon$. Then if $n > N$, we know $3/n < \epsilon$, and thus $3/n \in B_\epsilon(0)$. Since there are infinitely many odd $n > N$, and if $n > N$ is odd then $x_n = 3/n \in B_\epsilon(0)$, we see that 0 is an accumulation point for (x_n) by definition.

   Now we want to show that 2 is an accumulation point for x_n. For any $\epsilon > 0$ there is a N such that $1/N < \epsilon/2$ and thus $2/N < \epsilon$, and if $n > N$ then $2/n < \epsilon$. Thus $2 + 2/n \in B_\epsilon(2)$. There are infinitely many even $n > N$, and for any such n we have $x_n = 2 + 2/n \in B_\epsilon(2)$. Thus 2 is an accumulation point for (x_n) by definition.

6. If (x_n) is a bounded sequence of real numbers, prove that $\liminf x_n$ is an accumulation point.

   Solution: *Simple proof*: We've shown that $\liminf x_n$ is the limit of some subsequence (x_{n_k}) of the sequence (x_n). But the limit of any convergent subsequence is an accumulation point for the sequence.

   More explicit proof:

7. (a) Find a subset of $[0, 1]$ that doesn't have an accumulation point. Why does this not contradict the fact that $[0, 1]$ is compact?

   Solution: The set $\{0\}$ doesn't have an accumulation point, since it is finite.

   We proved that any infinite subset of a compact set has an accumulation point. But that doesn't mean any subset will have an accumulation point; any finite subset will have no accumulation points.

   (b) Find an infinite subset of $\mathbb{R}$ that doesn't have an accumulation point.

   Solution: The set $\mathbb{Z} \subset \mathbb{R}$ has no accumulation points: suppose x is an accumulation point of $\mathbb{Z}$. Then $B_{1/2}(x)$ must contain infinitely many points of $\mathbb{Z}$. But such a ball contains at most one point of $\mathbb{Z}$, which is a contradiction.

   Remark: Here we can find an infinite subset with no accumulation points, because $\mathbb{R}$ is not compact. We proved in the Bolzano-Weierstrass Theorem that any closed and bounded subset of $\mathbb{R}$ is compact. So if we have an infinite, bounded subset of $\mathbb{R}$, it is an infinite subset of some closed interval, and thus must have an accumulation point. But an unbounded infinite subset doesn't need to have an accumulation point.

8. Let (E, d) be a metric space, and let $V_1, \dots, V_n$ be a finite collection of (topologically) compact subsets. Prove that $V_1 \cup \dots \cup V_n$ is (topologically) compact.

   Solution:

   Suppose $V_1 \cup \dots \cup V_n \subset \bigcup U_\alpha$ for some collection of open sets U_α. We wish to show that $V_1 \cup \dots \cup V_n$ is a subset of the union of some finite collection of the U_α.

   For each i, we know that V_i is compact, and $V_i \subset \bigcup U_\alpha$. Thus there is some finite collection of subsets $U_{\alpha_{i,1}}, \dots, U_{\alpha_{i,m_i}}$ such that $V_i \subset \bigcup_{k=1}^{m_i} U_{\alpha_{i,m_i}}$.

   Then $V_1 \cup \dots \cup V_n \subset \bigcup_{i=1}^{n} \bigcup_{k=1}^{m_i} U_{\alpha_{i,m_i}}$, which is a finite union. Thus $V_1 \cup \dots \cup V_n$ is the union of some finite collection of the U_α. So by definition, $V_1 \cup \dots \cup V_n$ is (topologically) compact.