Math 310 Fall 2018
Real Analysis HW 8
Due Friday, November 2

You may *not* discuss the starred problem with classmates, though you should of course feel free to discuss it with me as much as you like. Linguistic precision is important for this problem. Submit this problem on a separate, detached sheet of paper.

$\star$ **Redo Problem:** If $f : E \to F$ is a continuous function and (x_n) is a convergent sequence in E such that $\lim_{n\to\infty} x_n = x$, then prove that $\lim_{n\to\infty} f(x_n) = f(x)$.

For the remainder of these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. Prove directly from the definition that $\lim_{x\to 2} 1/x = 1/2$.

2. Let $f(x,y) = \dfrac{x^2}{x^2+y^2}$ be a function $\mathbb{R}^2 \to \mathbb{R}$. Prove that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

3. Let $U \subset \mathbb{R}$ be an open interval containing a point a, (E,d) a metric space, and $f : U \to E$ a function. Define two functions

$$f_+ : U \cap \{x : x \geq a\} \to E \qquad\qquad f_- : U \cap \{x : x \leq a\} \to E$$
$$x \mapsto f(x) \qquad\qquad\qquad\qquad x \mapsto f(x)$$

   to be the restrictions of f to $U \cap \{x \geq a\}$ and $U \cap \{x \leq a\}$ respectively. Define

$$\lim_{x\to a^+} f(x) = \lim_{x\to a} f_+(x) \qquad \lim_{x\to a^-} f(x) = \lim_{x\to a} f_-(x)$$

   if those limits exist.

   Prove that $\lim_{x\to a} f(x)$ exists if and only if $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist and are equal.

4. Let $f(x) = \begin{cases} 1/q & x = p/q \text{ in lowest terms} \\ 0 & x \notin \mathbb{Q} \end{cases}$.

   Prove that f is continuous at a if and only if $a \notin \mathbb{Q}$. That is, prove that f is discontinuous at every rational number, but is continuous at every irrational number.