Math 310 Fall 2018
Real Analysis HW 8 Solutions
Due Friday, November 2

You may *not* discuss the starred problem with classmates, though you should of course feel free to discuss it with me as much as you like. Linguistic precision is important for this problem. Submit this problem on a separate, detached sheet of paper.

$\star$ **Redo Problem:** If $f : E \to F$ is a continuous function and (x_n) is a convergent sequence in E such that $\lim_{n\to\infty} x_n = x$, then prove that $\lim_{n\to\infty} f(x_n) = f(x)$.

For the remainder of these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. Prove directly from the definition that $\lim_{x\to 2} 1/x = 1/2$.

   Solution:

   Let $\epsilon > 0$ and let $\delta = \min\{2\epsilon, 1\}$. Then if $0 < |x-2| < \delta$, we can see that

   $$|2x| = 2|x-2+2| \geq 2(2-|x-2|) > 2(2-\delta) > 2.$$

   Therefore

   $$\left|\frac{1}{x} - \frac{1}{2}\right| = \frac{|x-2|}{|2x|} < \frac{\delta}{|2x|} < \frac{\delta}{2} < \epsilon.$$

2. Let $f(x,y) = \dfrac{x^2}{x^2+y^2}$ be a function $\mathbb{R}^2 \to \mathbb{R}$. Prove that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

   Solution: Let $(x_n, y_n) = (1/n, 0)$. Then $f(x_n,y_n) = f(1/n,0) = \frac{1/n^2}{1/n^2} = 1$, and thus $\lim_{n\to\infty} f(x_n,y_n) = 1$. So if $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, the limit must be 1.

   But now let $(a_n, b_n) = (0, 1/n)$. Then $f(a_n,b_n) = f(0,1/n) = \frac{0}{1/n^2} = 0$, and thus $\lim_{n\to\infty} f(a_n,b_n) = 0$. So if $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, the limit must be 0. This is a contradiction.

3. Let $U \subset \mathbb{R}$ be an open interval containing a point a, (E,d) a metric space, and $f : U \to E$ a function. Define two functions

   $$\begin{aligned} f_+ &: U \cap \{x : x \geq a\} \to E \qquad & f_- &: U \cap \{x : x \leq a\} \to E \\ x &\mapsto f(x) & x &\mapsto f(x) \end{aligned}$$

to be the restrictions of f to $U \cap \{x \geq a\}$ and $U \cap \{x \leq a\}$ respectively. Define

$$\lim_{x \to a^+} f(x) = \lim_{x \to a} f_+(x) \qquad \lim_{x \to a^-} f(x) = \lim_{x \to a} f_-(x)$$

if those limits exist.

Prove that $\lim_{x \to a} f(x)$ exists if and only if $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist and are equal.

Solution:

Suppose $\lim_{x \to a} f(x) = L$, and let $\epsilon > 0$. Then there is a $\delta > 0$ so that if $0 < |x-a| < \delta$ then $|f(x) - L| < \epsilon$.

Then if $x \in U \cap \{x \geq a\}$ such that $0 < |x-a| < \delta$, we know that $|f(x)-L| < \epsilon$, and thus $\lim_{x \to a} f_+(x) = L$, so by definition, $\lim_{x \to a^+} f(x) = L$. Similarly, if $x \in U \cap \{x \leq a\}$ such that $0 < |x-a| < \delta$, we know that $|f(x) - L| < \epsilon$, and thus $\lim_{x \to a} f_-(x) = L$, so by definition, $\lim_{x \to a^-} f(x) = L$.

Conversely, suppose $\lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$. Let $\epsilon > 0$. Then there is a $\delta_+ > 0$ so that if $x \in U \cap \{x \geq a\}$ and $0 < |x-a| < \delta_+$, then $|f_+(x) - L| = \epsilon$. And there is a $\delta_- > 0$ so that if $x \in U \cap \{x \leq a\}$ and $0 < |x-a| < \delta_-$, then $|f_-(x) - L| = \epsilon$.

So let $\delta = \min\{\delta_+, \delta_-\}$. Then if $0 < |x-a|\delta$, we know that $x \neq a$, so either $x < a$ or $x > a$. If $x < a$, we know that $x \in U \cap \{x \leq a\}$, so since $|x-a| < \delta \leq \delta_-$, we know that $|f(x) - L| < \epsilon$. And if $x > a$, we know that $x \in U \cap \{x \geq a\}$, so since $|x-a| < \delta \leq \delta_+$, we know that $|f(x) - L| < \epsilon$.

Either way, we have that $|f(x) - L| < \epsilon$, and thus $\lim_{x \to a} f(x) = L$.

4. Let $f(x) = \begin{cases} 1/q & x = p/q \text{ in lowest terms} \\ 0 & x \notin \mathbb{Q} \end{cases}$.

Prove that f is continuous at a if and only if $a \notin \mathbb{Q}$. That is, prove that f is discontinuous at every rational number, but is continuous at every irrational number.

Solution: First, suppose $a = p/q \in \mathbb{Q}$. Then $f(a) = 1/q$. Let $\epsilon = 1/q$, and let $\delta > 0$. Then there is some irrational number $x_0 \in (a - \delta, a + \delta)$. (We proved this in class: Let b be any rational number, and let $N > |b|/\delta$. Then $|b|/N < \delta$, so $a + |b|/N$ is irrational and lies in $(a - \delta, a + \delta)$. Let $x_0 = a + |b|/N$.)

Then $|x_0 - a| < \delta$, but $|f(x_0) - 1/q| = 1/q \geq \epsilon$. Thus $\lim_{x \to a} f(x) \neq 1/q$, so f is not continuous at a.

Now suppose a is irrational. We want to show that $\lim_{x \to a} f(x) = 0 = f(a)$. Let $\epsilon > 0$. Then there is a $N > 1/\epsilon$.

Let $\delta = \min\{d(a, p/q) : 0 < q \leq N, d(a, p/q) < 1\}$. We know this minimum exists because the set is finite: within distance 1 of a, there are at most two points with denominator 1, four points with denominator 2, six points with denominator 3, and so on; there are finitely many q with $0 < q \leq N$.

Now if $|x - a| < \delta$, there are two possibilities. If x is irrational, then $f(x) = 0$, so $|f(x) - f(a)| = |0 - 0| = 0 < \epsilon$. If $x = p/q$ is rational, then we know that $q > N$, and thus $|f(x) - 0| = 1/q < 1/N < \epsilon$. Thus we have shown that $\lim_{x \to a} f(x) = 0 = f(a)$; and so f is continuous at a for any irrational a.