Math 310 Fall 2018
Real Analysis HW 9 Solutions
Due Monday, November 12

You may *not* discuss the starred problem with classmates, though you should of course feel free to discuss it with me as much as you like. Linguistic precision is important for this problem. Submit this problem on a separate, detached sheet of paper.

$\star$ **Redo Problem:** Let $f : E \to F$ be a bijection, and E compact. Prove that the inverse function $f^{-1} : F \to E$ is continuous. (Hint: prove that $f(V)$ is closed for every closed set V, and thus that $f(U)$ is open for every open set U).

For the remainder of these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. (a) Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a sequence (x_n) of real numbers so that $(f(x_n))$ converges but (x_n) does not.

   (b) Find a continuous function $f : V \to \mathbb{R}$ where V is a closed subset of $\mathbb{R}$ that doesn't attain a maximum value.

   (c) Find a continuous function $f : S \to \mathbb{R}$ where S is a closed and bounded subset of some metric space that doesn't attain a maximum value.

2. Let E, F be metric spaces, and $S \subset E$. Define $\chi_S : E \to F$ by $\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$. Show that χ_S is continuous at a if and only if a is not on the boundary of S.

3. If $f : E \to F$ is a function on metric spaces and E is the discrete metric space, prove that f is continuous.

4. If $f : E \to F$ and $g : F \to G$ are continuous functions of metric spaces, prove that $g \circ f$ is continuous. (Hint: use the topological result about open sets, not the limit definition).

5. Let $p_i : \mathbb{R}^n \to \mathbb{R}$ be the projection into the $i$the coordinate given by $p_i(x_1, \ldots, x_n) = x_i$. Prove that p_i is a continuous function.

6. (a) Let (E, d) be a metric space, let S be a non-empty compact subset, and let $x \in E$. Show that the "distance from x to S" given by $d(x, S) = \min\{d(x, y) : y \in S\}$ exists.

   (b) Find a non-empty subset $S \subset \mathbb{R}$ and a point $x \in \mathbb{R}$ such that $d(x, S)$ does not exist.

7. Let (E, d) be a metric space, $S \subset E$, and $f : S \to \mathbb{R}$ be a function, and suppose $\lim_{x \to a} f(x)$ exists. Suppose there are $r, s \in \mathbb{R}$ such that $r \leq f(x) \leq s$ for all $x \in S$. Prove that $r \leq \lim_{x \to a} f(x) \leq s$.

8. Suppose $f : [0, 1] \to [0, 1]$ is continuous. Show that f has a fixed point. That is, show there is an $x \in [0, 1]$ such that $f(x) = x$.