Math 310 Fall 2018
Real Analysis HW 9 Solutions
Due Monday, November 12

You may *not* discuss the starred problem with classmates, though you should of course feel free to discuss it with me as much as you like. Linguistic precision is important for this problem. Submit this problem on a separate, detached sheet of paper.

$\star$ **Redo Problem:** Let $f : E \to F$ be a continuous bijection, and E compact. Prove that the inverse function $f^{-1} : F \to E$ is continuous. (Hint: prove that $f(V)$ is closed for every closed set V, and thus that $f(U)$ is open for every open set U).

For the remainder of these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. (a) Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a sequence (x_n) of real numbers so that $(f(x_n))$ converges but (x_n) does not.

   (b) Find a continuous function $f : V \to \mathbb{R}$ where V is a closed subset of $\mathbb{R}$ that doesn't attain a maximum value.

   (c) Find a continuous function $f : S \to \mathbb{R}$ where S is a closed and bounded subset of some metric space that doesn't attain a maximum value.

   Solution: For all of these problems there are many possible solutions. I will provide a sample solution for each problem.

   (a) Let $x_n = (-1)^n$ so that $(x_n) = -1, 1, -1, 1, \ldots$, and let $f(x) = |x|$. Then $f(x_n) = 1$ so $\lim_{n\to\infty} f(x_n) = 1$, but $\lim_{n\to\infty} x_n$ does not exist.

   (b) Any solution here will require V to be unbounded, since a closed and bounded subset of $\mathbb{R}$ is compact.

   Let $V = \mathbb{R}$, which is closed, and define $f(x) = x$. Then f attains no maximum value by the Archimedean property.

   (c) No example set in $\mathbb{R}^n$ as the metric space can possibly work, since a closed and bounded subset of $\mathbb{R}^n$ is compact.

   We can take E to be the set $\mathbb{R}$ under the discrete metric, and $f : E \to \mathbb{R}$ be defined by $f(x) = x$. Then clearly f attains no maximum value. But E is closed and bounded since every set is closed and bounded in the discrete metric. (We could also take $S = (0, 1)$ and get the same result).

2. Let E, F be metric spaces, and $S \subset E$. Define $\chi_S : E \to F$ by $\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$. Show that χ_S is continuous at a if and only if a is not on the boundary of S.

   Solution: Suppose $a \in \partial S$. Then for each $n \in \mathbb{N}$ there is a point $x_n \in B_{1/n}(a) \cap S$ and a point $y_n \in B_{1/n}(a) \cap S^C$. Then by construction, $\lim_{n\to\infty} x_n = a = \lim_{n\to\infty} y_n$.

   For each x_n we have $\chi_S(x_n) = 1$, so $\lim_{n\to\infty} \chi_S(x_n) = 1$; thus if $\lim_{x\to a} \chi_S(x)$ exists it must be 1. But for each y_n we have $\chi_S(y_n) = 0$, so $\lim_{n\to\infty} \chi_S(y_n) = 0$; thus if $\lim_{x\to a} \chi_S(x)$ exists it must be 0. But this is a contradiction, so $\lim_{x\to a} \chi_S(x)$ does not exist, and $\chi_S(x)$ is not continuous at a.

   Conversely, we need to consider the cases where $a \in \mathring{S}$ or $a \in \mathring{S^C}$. If $a \in \mathring{S}$ then there exists some $\delta > 0$ so that $B_\delta(a) \subset \mathring{S}$. Then for every $\epsilon > 0$, if $d(x, a) < \delta$ then $x \in \mathring{S}$ and thus $\chi_S(x) = 1$, and so $|1 - \chi_S(x)| = 0 < \epsilon$. Thus $\lim_{x\to a} \chi_S(x) = 1 = \chi_S(a)$.

   An identical argument shows that if $a \in \mathring{S^C}$ then $\lim_{x\to a} \chi_S(x) = 0 = \chi_S(a)$.

3. If $f : E \to F$ is a function on metric spaces and E is the discrete metric space, prove that f is continuous.

   Solution: The easiest proof is probably the topological proof. Suppose $U \subset F$ is open. Then $f^{-1}(U) \subset E$ is open because any subset of a discrete metric space is open; thus f is continuous.

   We could prove this in two other ways. One is to claim that E has no accumulation points, and thus it is continuous at every accumulation point. Or we can say that if $a \in E$, then for every $\epsilon > 0$, if $\delta < 1$ and $d(x, a) < \delta$ then $x = a$ so $|f(x) - f(a)| = 0 < \epsilon$, and thus f is continuous at a.

4. If $f : E \to F$ and $g : F \to G$ are continuous functions of metric spaces, prove that $g \circ f$ is continuous. (Hint: use the topological result about open sets, not the limit definition).

   Solution: Let $U \subset G$ be open. Then $g^{-1}(U)$ is open since g is continuous. But f is continuous, so $f^{-1}(g^{-1}(U))$ is open. Thus $f^{-1}(g^{-1}(U))$ is open. But $f^{-1}(g^{-1}(U))$ is the same as $(f \circ g)^{-1}(U)$, so the preimage of an open set under $f \circ g$ is open. Thus $f \circ g$ is continuous.

5. Let $p_i : \mathbb{R}^n \to \mathbb{R}$ be the projection into the $i$the coordinate given by $p_i(x_1, \dots, x_n) = x_i$. Prove that p_i is a continuous function.

   Solution: Let $\vec{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$; we claim that p_i is continuous at $\vec{a}$.

   Let $\epsilon > 0$, and set $\delta = \epsilon$. Then if $d(\vec{x}, \vec{a}) < \delta$, we know that $\max\{|x_j - a_j|\} < \delta$, so in particular $|p_i(\vec{x}) - p_i(\vec{a})| = |x_i - a_i| < \delta = \epsilon$. Thus $\lim_{\vec{x}\to\vec{a}} p_i(\vec{x}) = \vec{a}$, so p_i is continuous at $\vec{a}$.

6. (a) Let (E, d) be a metric space, let S be a non-empty compact subset, and let $x \in E$. Show that the "distance from x to S" given by $d(x, S) = \min\{d(x, y) : y \in S\}$ exists.

   (b) Find a non-empty subset $S \subset \mathbb{R}$ and a point $x \in \mathbb{R}$ such that $d(x, S)$ does not exist.

   Solution:

(a) Define $f : S \to \mathbb{R}$ by $f(y) = d(x, y)$. We claim that f is continuous. Fix $z \in S$ and suppose $\epsilon > 0$, and let $\delta = \epsilon$. Then if $d(y, z) < \delta$ we have

$$|f(y) - f(z)| = |d(x, y) - d(x, z)| \leq d(y, z) < \delta = \epsilon$$

and thus $\lim_{y \to z} f(y) = f(z)$. Thus f is continuous at z.

Then f is a continuous function from a compact set to $\mathbb{R}$, and so by the extreme value theorem it achieves a minimum value. Thus $d(x, S) = \min\{d(x, y) : y \in S\} = \min\{f(y) : y \in S\}$ exists.

(b) Let $S = (0, 1)$ and $x = 0$. Then $\{d(x, y) : y \in S\}$ has no minimum value, since for any $r > 0$ there exists a y with $0 < y < r$ so that $d(x, y) = y < r$; but there is no $y \in S$ with $d(x, y) = 0$ since $0 \notin S$.

7. Let (E, d) be a metric space, $S \subset E$, and $f : S \to \mathbb{R}$ be a function, and suppose $\lim_{x \to a} f(x)$ exists. Suppose there are $r, s \in \mathbb{R}$ such that $r \leq f(x) \leq s$ for all $x \in S$. Prove that $r \leq \lim_{x \to a} f(x) \leq s$.

   Solution: Let x_n be a sequence in E such that $\lim_{n \to \infty} x_n = a$. Then we know that $\lim_{n \to \infty} f(x_n) = f(a)$ exists. But $r \leq f(x_n) \leq s$ for each x_n, so $r \leq \lim_{n \to \infty} f(x_n) \leq s$ and thus $r \leq \lim_{x \to a} f(x) \leq s$.

8. Suppose $f : [0, 1] \to [0, 1]$ is continuous. Show that f has a fixed point. That is, show there is an $x \in [0, 1]$ such that $f(x) = x$.

   Solution: Define $g : [0, 1] \to \mathbb{R}$ by $g(x) = f(x) - x$. Then g is a difference of continuous functions, and thus continuous. We wish to prove that there is a $c \in [0, 1]$ such that $g(c) = 0$; in that case we have $f(c) - c = 0$ and hence $f(c) = c$.

   We see that $g(0) = f(0) - 0 \geq 0$, and $g(1) = f(1) - 1 \leq 0$. If either $g(0) = 0$ or $g(1) = 0$ then we are done, so assume $g(0) > 0$ and $g(1) < 0$. Then by the intermediate value theorem there is a $c \in (0, 1)$ such that $g(c) = 0$.