Math 310 Exam 1 Solutions

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Problem 1. Prove directly from the definition that $\lim_{n\to\infty} \frac{3}{n} = 0$.

Solution: Let $\epsilon > 0$ and set $N > 3/\epsilon$. Then if $n < N$, we have

$$\left|\frac{3}{n} - 0\right| = \frac{3}{n} < \frac{3}{N} < \epsilon.$$

Problem 2. Define $\ell_\infty(\mathbb{R})$ to be the set of bounded sequences of real numbers, and define $d(x_n, y_n) = \sup\{|x_n - y_n| : n \in \mathbb{N}\}$. Prove that this is a metric.

Solution:

1. Every element of $\{|x_n - y_n| : n \in \mathbb{N}\}$ is non-negative, so the supremum is non-negative.
   If $d(x_n, y_n) = 0$, this means that $\sup\{|x_n - y_n|\} = 0$. Thus $|x_n - y_n| = 0$ for each n, so $(x_n) = (y_n)$.
2. $|x_n - y_n| = |y_n - x_n|$, so
   $$d(x_n, y_n) = \sup\{|x_n - y_n|\} = \sup\{|y_n - x_n|\} = d(y_n, x_n).$$
3. For each n, we know that $|x_n - z_n| \leq |x_n - y_n| + |y_n - z_n|$. Thus
   $$\begin{aligned} d(x_n, z_n) &= \sup\{|x_n - z_n|\} \leq \sup\{|x_n - y_n| + |y_n - z_n|\} \\ &\leq \sup\{|x_n - y_n\} + \sup\{|y_n - z_n|\} = d(x_n, y_n) + d(y_n, z_n). \end{aligned}$$

Problem 3. Let $S \subset \mathbb{R}^2$ be defined by $S = \{(x, y) : y > 3\}$. Prove that S is open in one of the sup, sum, or Euclidean metrics.

Solution:
Let $(x, y) \in S$. Then $y > 3$; set $r = y - 3$. We claim that $B_r(x, y) \subset S$ in the sup metric.
Suppose $(x_1, y_1) \in B_r(x, y)$. Then

$$\begin{aligned} d((x, y), (x_1, y_1)) &< r \\ \sup\{|x - x_1|, |y - y_1|\} &< r \\ |y - y_1| &< r = y - 3 \\ 3 - y < y_1 - y &< y - 3 \\ 3 < y_1 &< 2y - 3. \end{aligned}$$

Thus $3 < y_1$ and so $(x_1, y_1) \in S$.
Thus $B_r(x, y) \subset S$ and so S is open.

Problem 4. Let (x_n) be a sequence of real numbers. Prove that $\lim_{n\to\infty} x_n = 0$ if and only if $\lim_{n\to\infty} |x_n| = 0$.

Solution:
Suppose $\lim_{n\to\infty} x_n = 0$. Let $\epsilon > 0$. There is some $N \in \mathbb{N}$ so that if $n > N$ then $|x_n - 0| < \epsilon$. Then if $n > N$, we have

$$||x_n| - 0| = ||x_n|| = |x_n| = |x_n - 0| < \epsilon$$

so $\lim_{n\to\infty} |x_n| = 0$.

Conversely, suppose $\lim_{n\to\infty} |x_n| = 0$. Let $\epsilon > 0$. There is some $N \in \mathbb{N}$ so that if $n > N$ then $||x_n| - 0| < \epsilon$. Then if $n > N$, we have

$$|x_n - 0| = |x_n| = |x_n| - 0 = ||x_n| - 0| < \epsilon$$

so $\lim_{n\to\infty} x_n = 0$.

Problem 5. Let (x_n) be a convergent sequence of real numbers, so that $\lim\limits_{n\to\infty} x_n = x$. Prove that the set $\{x_n\}$ is bounded above.

Solution:

Since $\lim_{n\to\infty} x_n = x$, there is some $N \in \mathbb{N}$ so that $|x_n - x| < 1$ when $n > N$. Thus for all $n > N$, we know that $x_n - x < 1$ and thus $x_n < x + 1$.

The set $\{x_1, x_2, \ldots, x_N, x + 1\}$ is finite, so it has some maximum. Let $M = \max\{x_1, \ldots, x_N, x + 1\}$. We claim M is an upper bound for $\{x_n\}$. For if $n \leq N$, then $x_n \leq M$ by definition. And if $n > N$, then $x_n < x + 1 \leq M$. Thus $x_n \leq M$ for all $n \in \mathbb{N}$, and so M is an upper bound for $\{x_n\}$.