Math 310 Exam 2 Solutions

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Problem 1. Let $f(x) = 4x + 2$. Prove directly from the definition that $\lim_{x \to 2} f(x) = 10$.

Solution: Let $\epsilon > 0$ and set $\delta = \epsilon/4$. Then if $|x - 2| < \delta$, we have

$$|f(x) - 10| = |4x - 8| = 4|x - 2| < 4\delta = \epsilon.$$

Thus by definition, $\lim_{x \to 2} 4x + 2 = 10$.

Problem 2. Prove that any metric space with finitely many points is sequentially compact.

Solution: Let E be a finite metric space, and let (x_n) be a sequence of points in E. Then by the pigeonhole principle, at least one point of E must occur in (x_n) infinitely many times. Thus (x_n) has a constant subsequence, which is convergent.

Thus any sequence in E has a convergent subsequence, and so E is sequentially compact.

Alternate proof: E is finite, so it has no infinite subet. Thus every infinite subset contains all its accumulation points; we have shown that this implies that E is compact.

Problem 3. Suppose $f : [0, 1] \to (0, 1)$ is a bijection. Prove that f is not continuous.

Solution: Suppose f is continuous. Then since f is a continuous function on a compact set, it has a maximum value $y \in (0, 1)$. But $y < 1$, so there is a number z such that $y < z < 1$, and thus $z \notin f([0, 1])$, contradicting the fact that f is a bijection.

Note: It's not enough to observe that $f^{-1}(0, 1) = [0, 1]$, because $[0, 1]$ is open in *itself*. The other option is to show that a continuous bijection $f : [0, 1] \to (0, 1)$ gives you an extension to a continuous map $f : \mathbb{R} \to (0, 1)$ or something, in which case we know that $f^{-1}(0, 1) = [0, 1]$ isn't open in $\mathbb{R}$. (And you can in fact do that, by just making $f(x) = f(1)$ for $x > 1$ or something like that; but that does involve a bit of extra work).

Problem 4. Prove that any sequentially compact metric space is complete.

Solution:

Let (E, d) be a compact metric space, and let (x_n) be a Cauchy sequence. Then (x_n) has a convergent subsequence (x_{n_k}) which converges to some point $x \in E$. We claim that $\lim_{n \to \infty} x_n = x$.

We know from a homework excersice that if a Cauchy sequence has a convergent subsequence, then it converges. But we can also prove it if we want.

Let $\epsilon > 0$. There there is a $N_1 \in \mathbb{N}$ so that $d(x_n, x_m) < \epsilon/2$ if $n, m > N_1$. And there is a $N_2 \in \mathbb{N}$ such that $d(x_{n_k}, x) < \epsilon/2$ if $k > N_2$. Let $N = \max\{N_1, N_2\}$. Then if $m > N$, we know that $d(x_m, x_{n_m}) < \epsilon/2$ and $d(x_{n_m}, x) < \epsilon/2$, so by the triangle inequality, $d(x_m, x) < \epsilon/2 + \epsilon/2 = \epsilon$. Thus $\lim_{n \to \infty} x_n = x$.

Thus every Cauchy sequence converges, and so E is complete.

Problem 5. (a) Suppose $f : E \to \mathbb{R}$ is a function of metric spaces, and $a \in E$ such that $\lim_{x \to a} f(x) = L$. Show that $\lim_{x \to a} |f(x)| = |L|$.

(b) Find a function $g : \mathbb{R} \to \mathbb{R}$ and a point $a \in \mathbb{R}$ such that $\lim_{x\to a} |g(x)|$ exists but $\lim_{x\to a} g(x)$ does not.

Solution:

(a) Let $\epsilon > 0$. Then there is a δ so that if $|x - a| < \delta$, then $|f(x) - L| = \epsilon$.

Then if $|x - a| < \delta$, by the reverse triangle inequality we have $||f(x)| - |L|| \leq |f(x) - L| < \epsilon$.

Thus by definition, $\lim_{x\to a} |f(x)| = L$.

(b) Let $g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$ Then $|g(x)| = 1$ so $\lim_{x\to 0} |g(x)| = 1$. But $\lim_{x\to 0^+} g(x) = 1$ and $\lim_{x\to 0^-} g(x) = -1$ so $\lim_{x\to 0} g(x)$ does not exist.