Week 7: Elliptic Curves

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Occidental College

October 9, 2018

It is possible to write endlessly on elliptic curves.

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Serge Lang

An example of the infinite dihedral group. We can accomplish any symmetry by combining a translation of some number of units with a possible 180° rotation.

An *elliptic curve* is:

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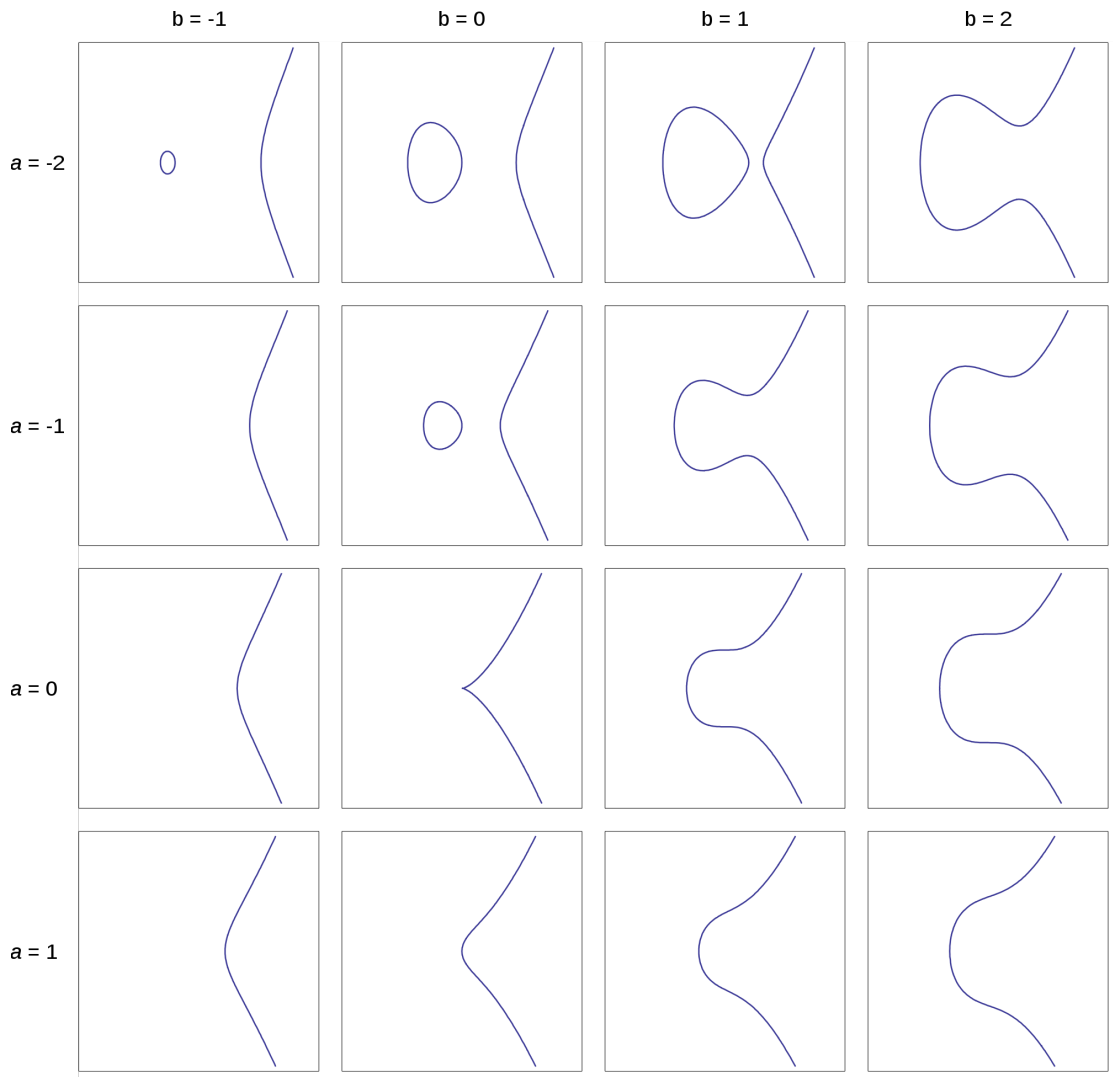
- *NOT* an ellipse!



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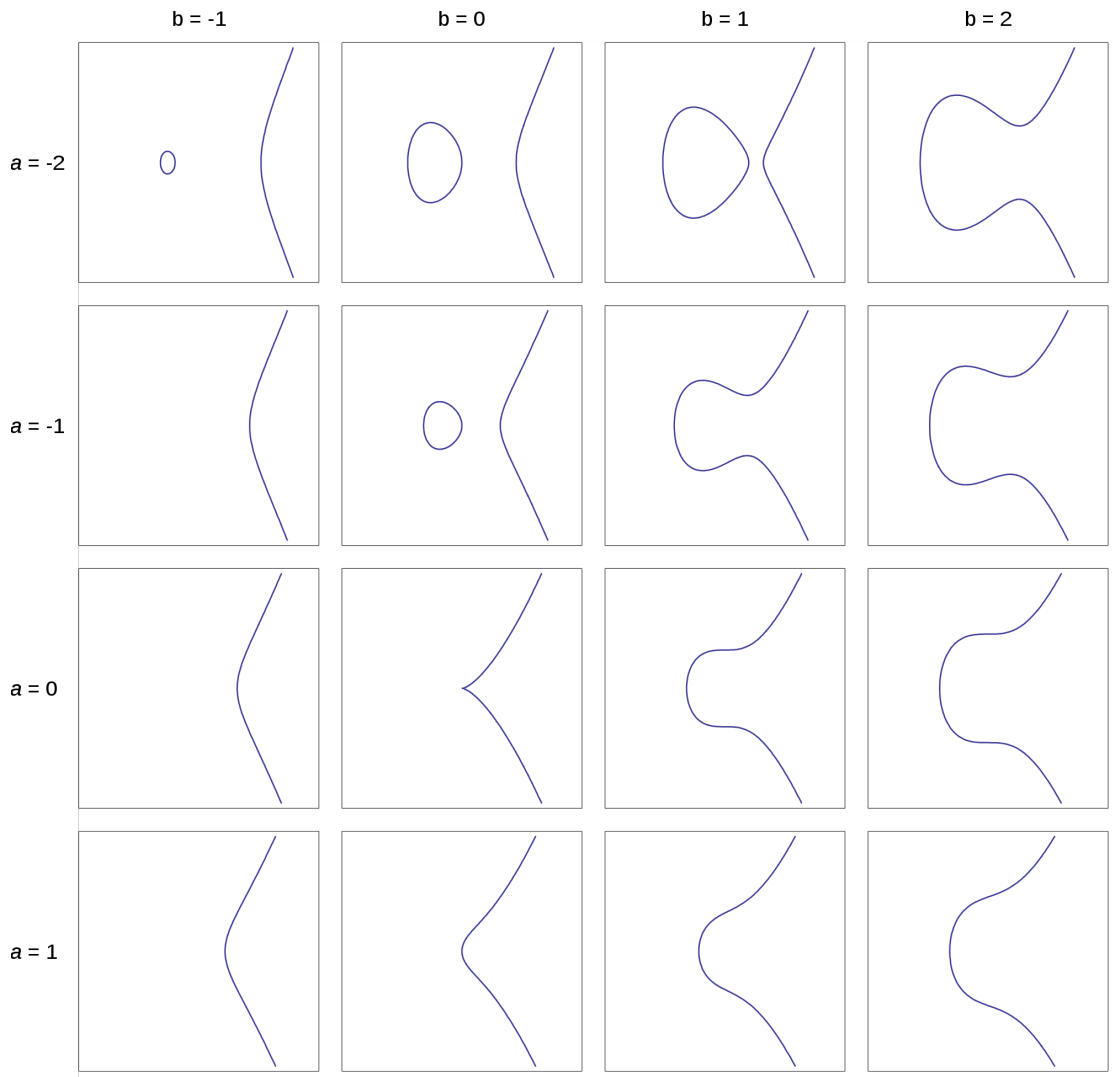

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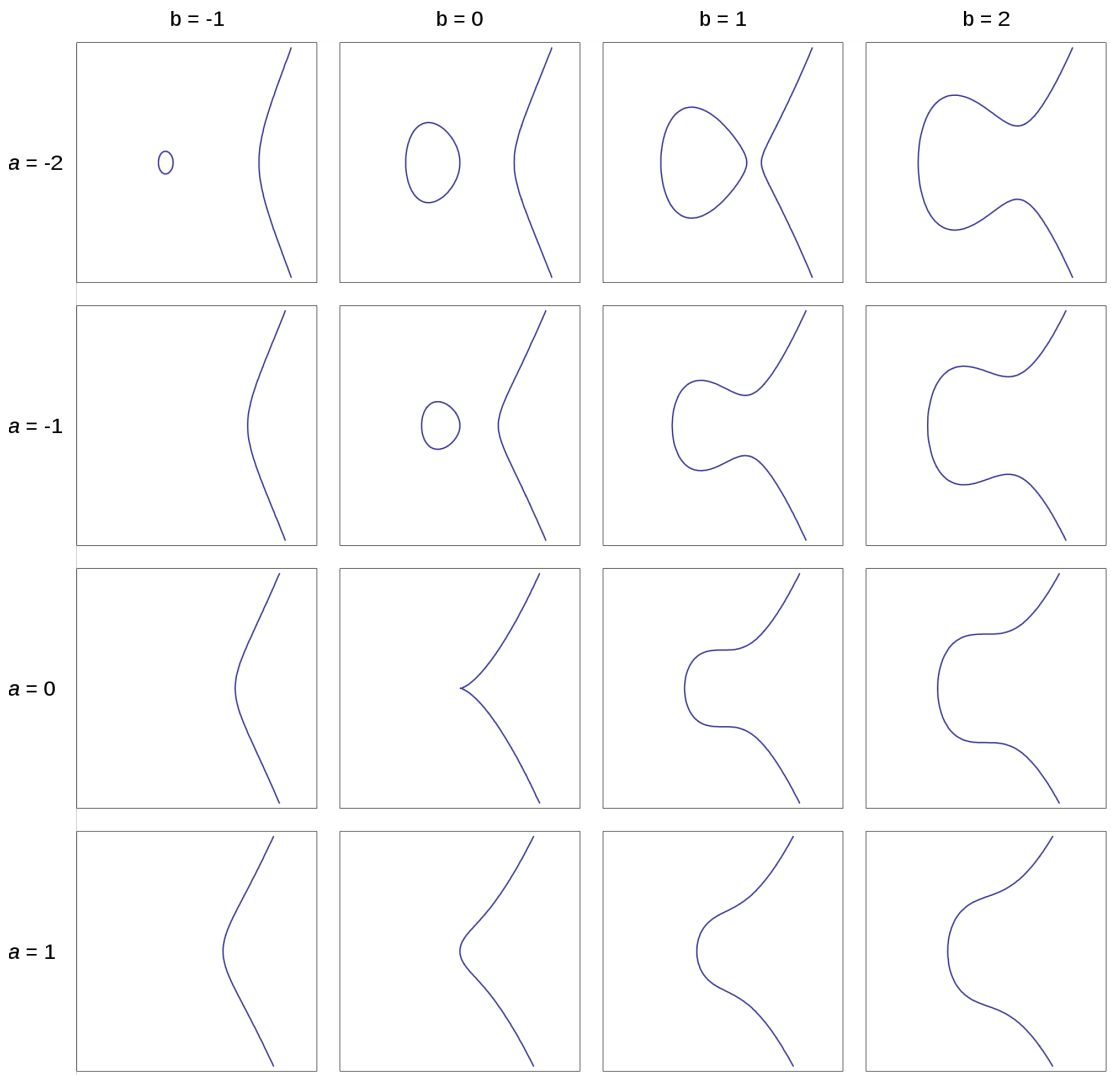
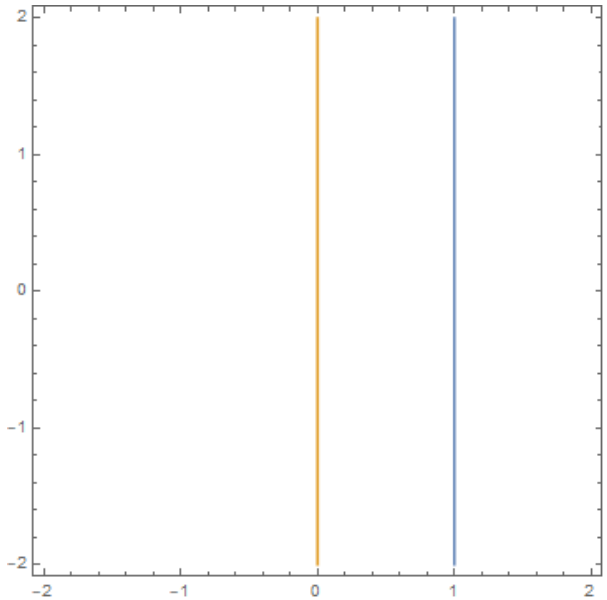
Key Question

How many rational points are there?

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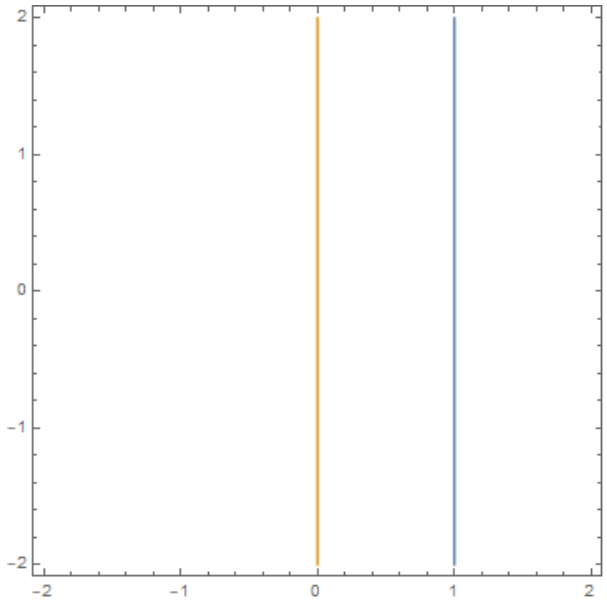
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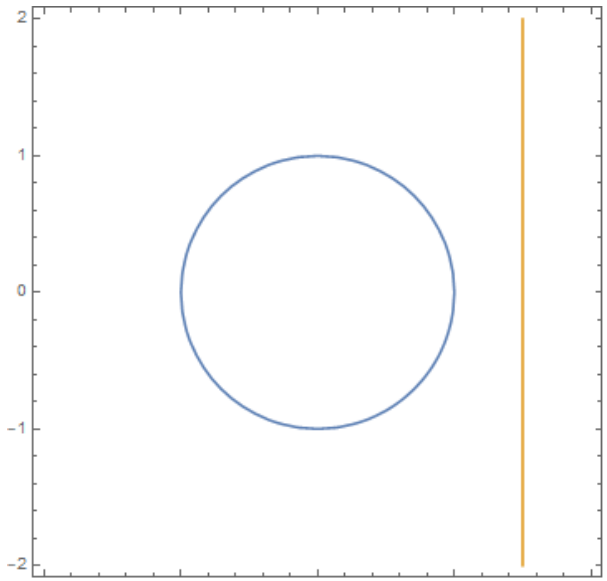
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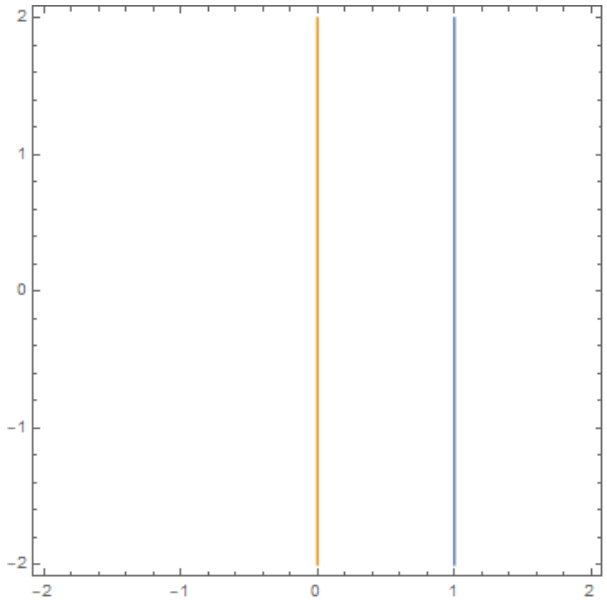
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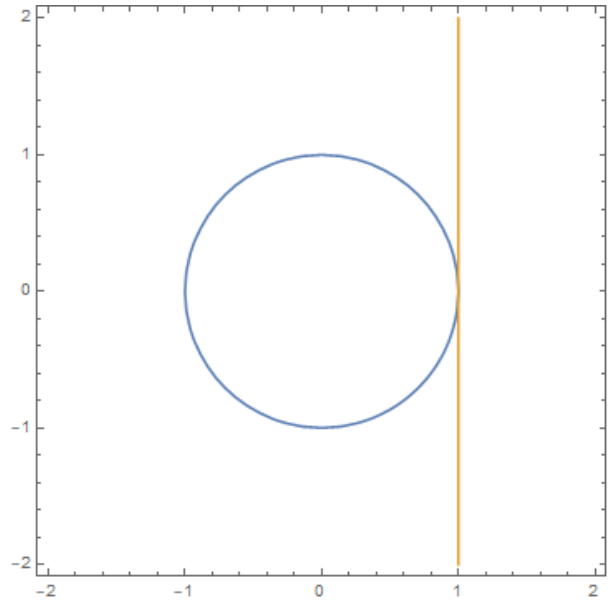
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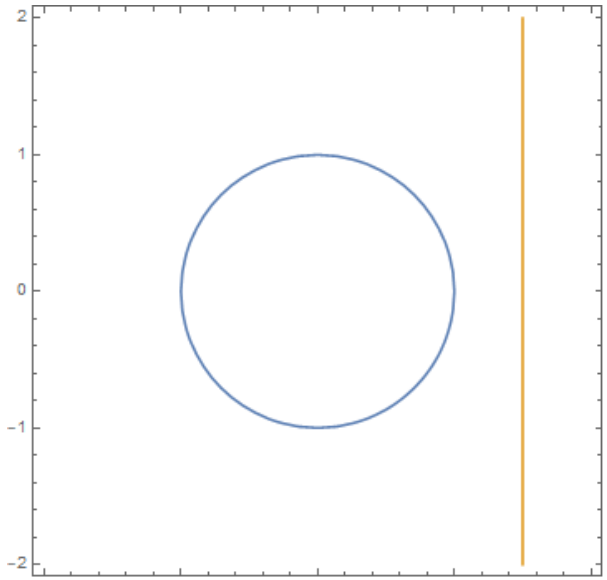
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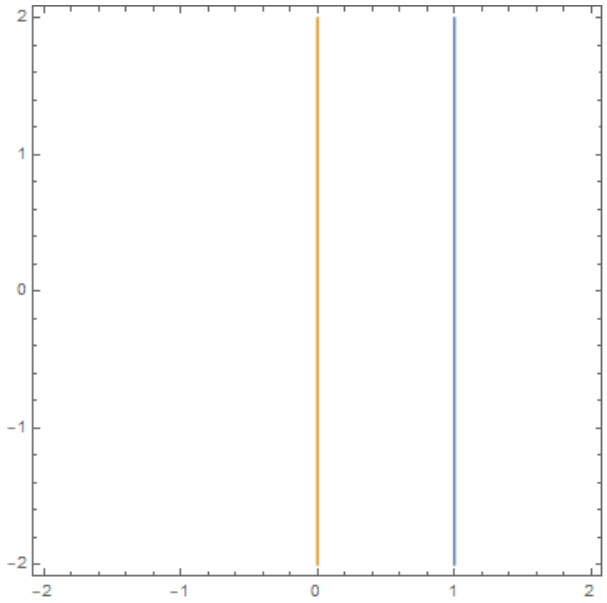
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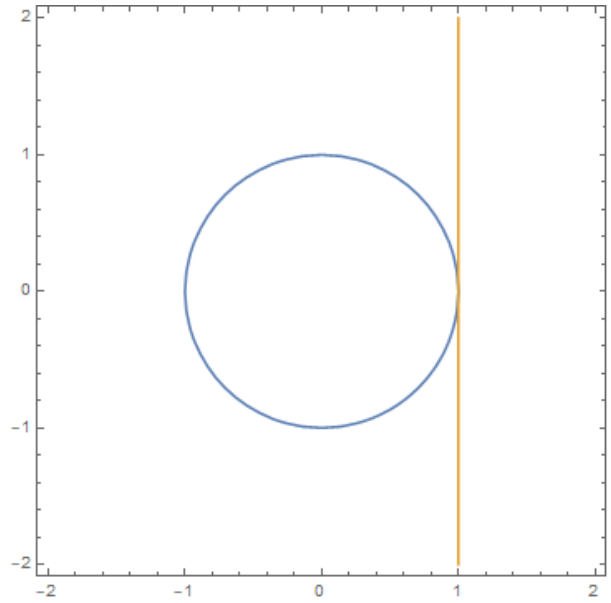
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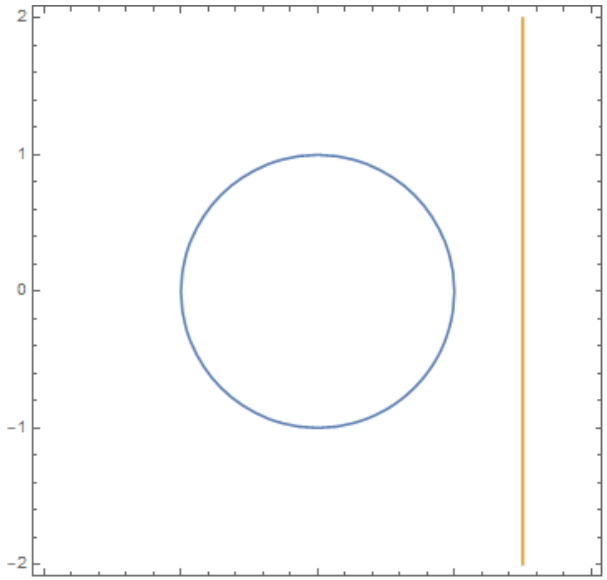
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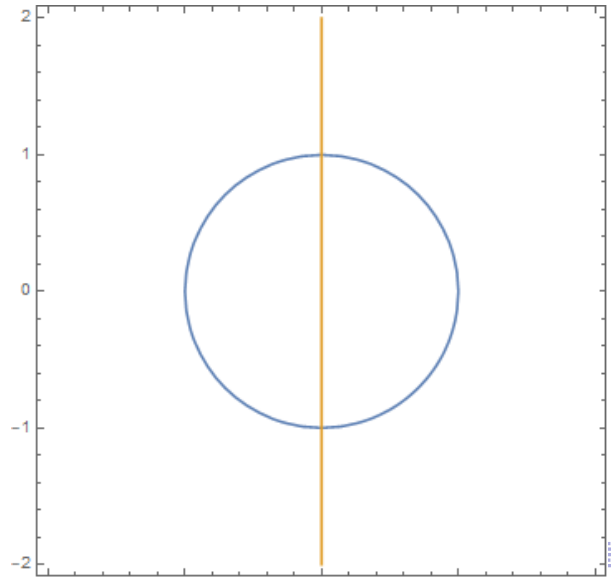
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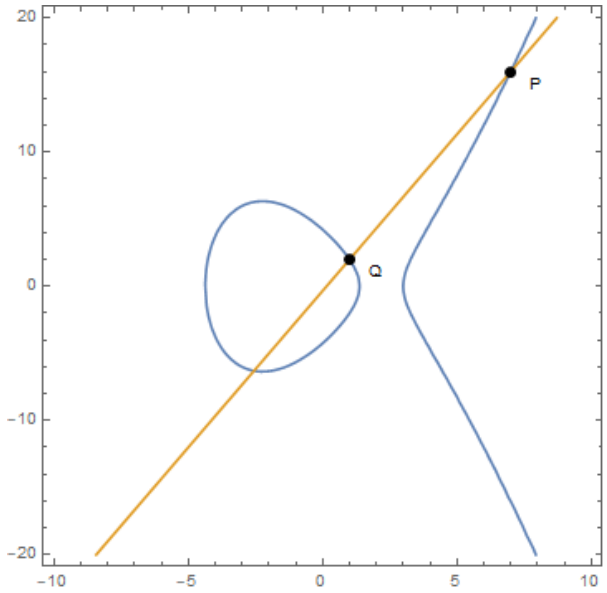
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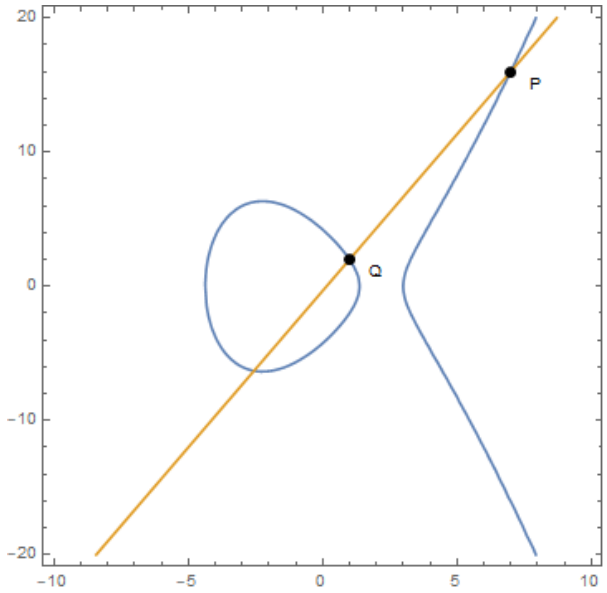
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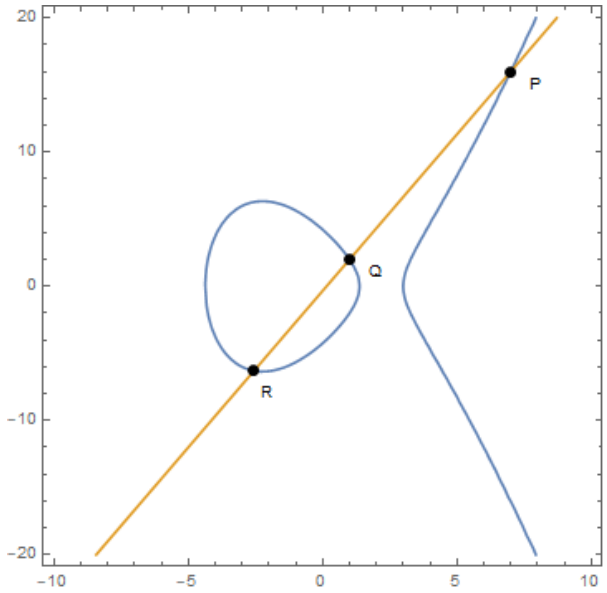
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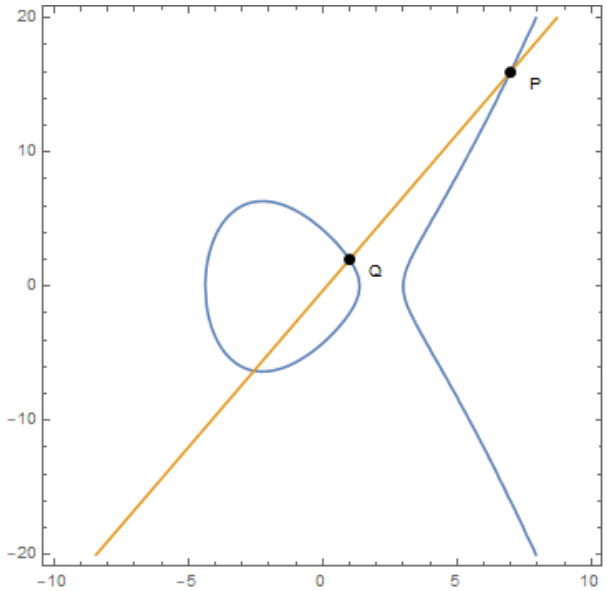
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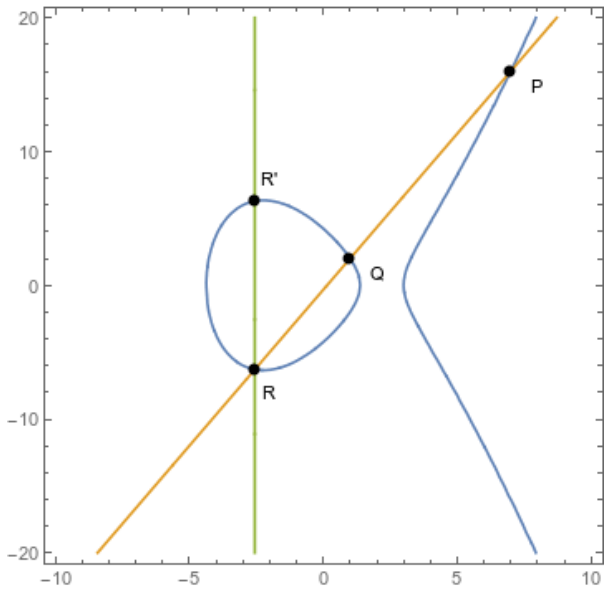
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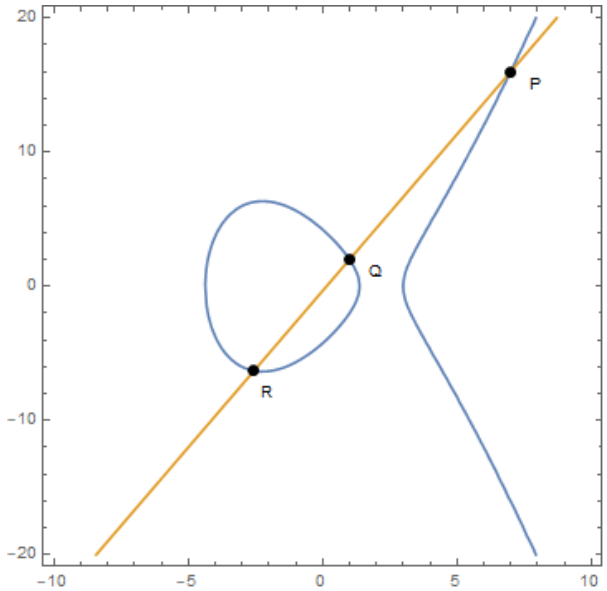
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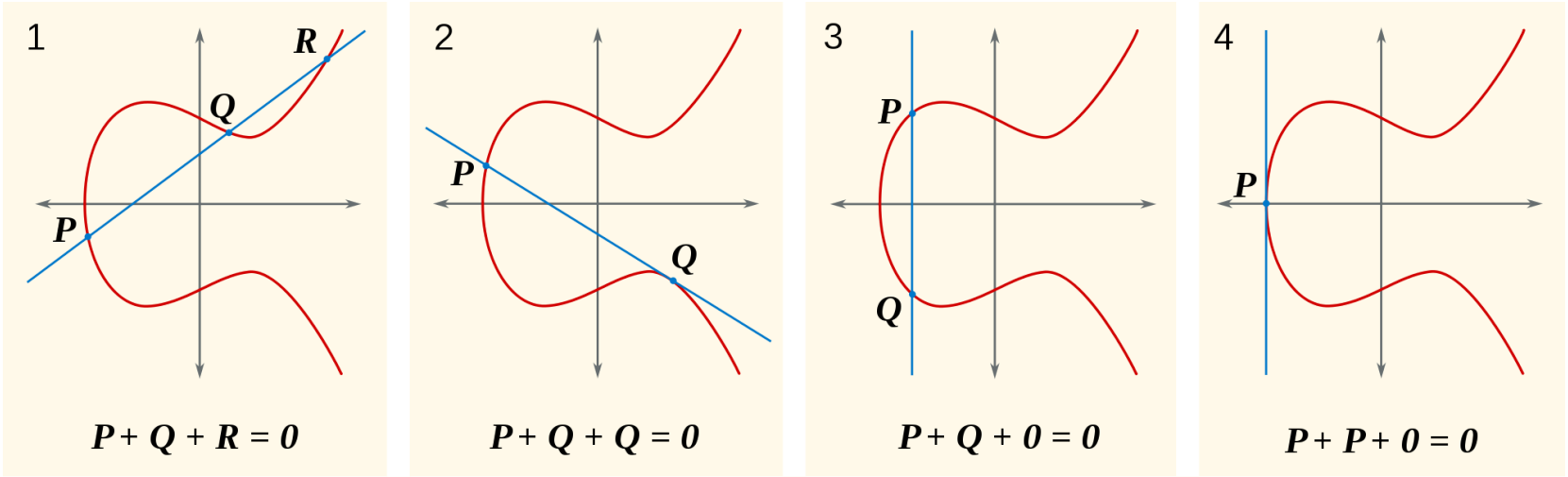
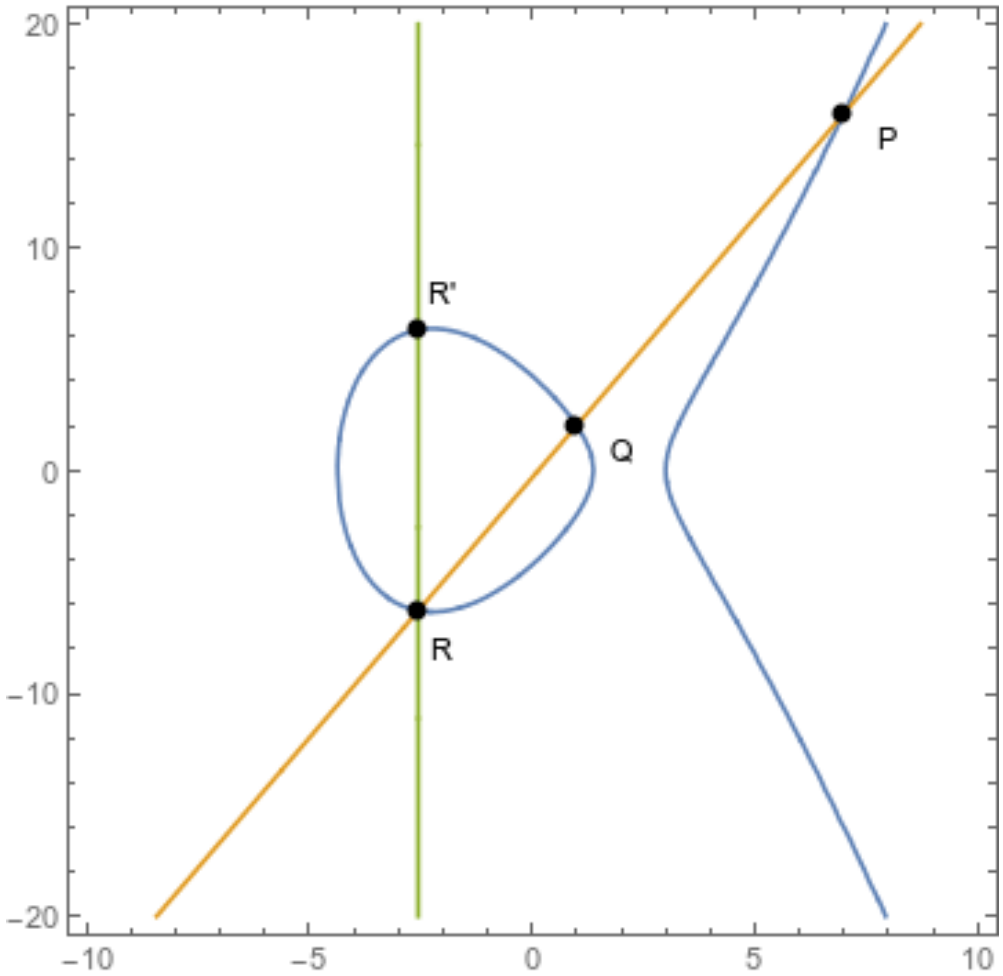


Figure: The group law on elliptic curves
Emmanuel Boutet / CC-BY-SA-3.0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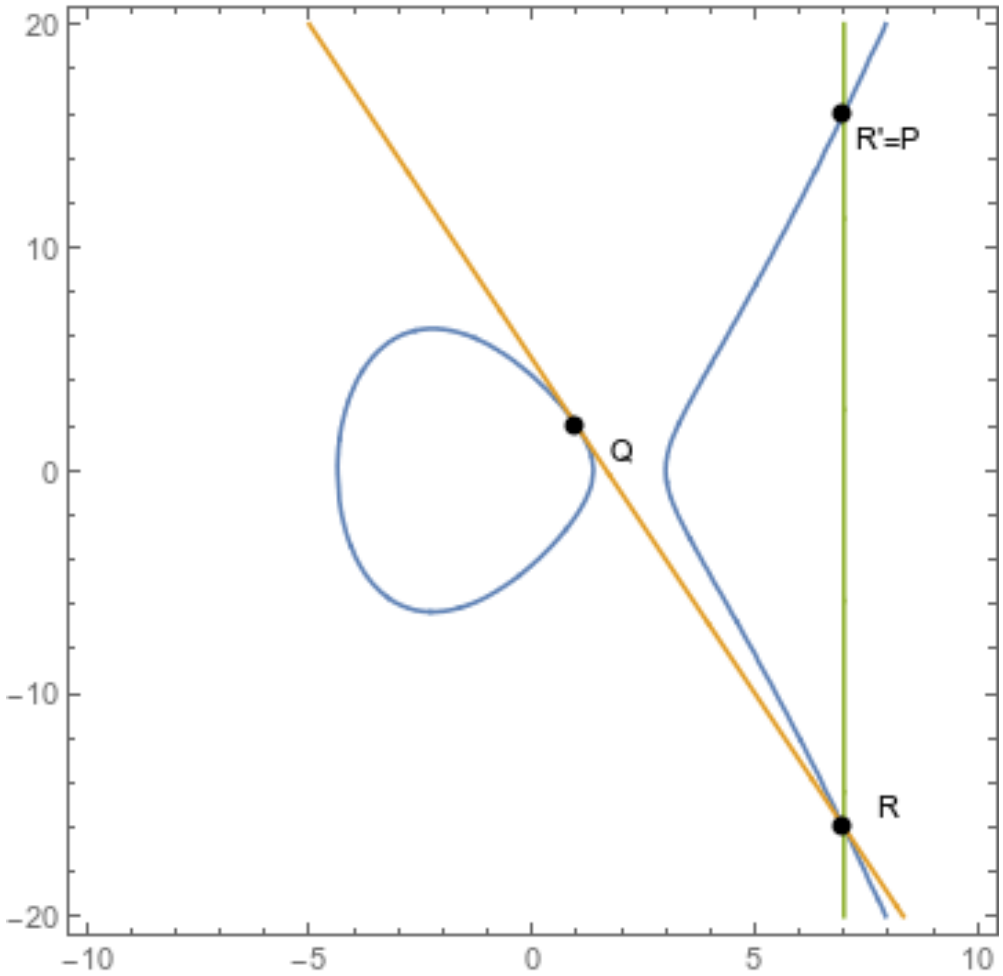
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