

## 4 Continuity

### 4.1 Continuous Functions

**Definition 4.1.** A function  $f$  is *continuous* at a number  $a$  if

$$\lim_{x \rightarrow a} f(x) = f(a).$$

Intuitively, this means that the function doesn't "jump" at  $a$ ; as  $x$  gets closer to  $a$  the value of  $f(x)$  gets closer to the value of  $f(a)$ .

Note that this definition requires three things to be true:

1. The function is defined at  $a$ ; that is,  $a$  is in the domain of  $f$ .
2.  $\lim_{x \rightarrow a} f(x)$  exists.
3. The two numbers are the same.

There are a few different ways for a function to be discontinuous at a point:

1. A function  $f$  has a *removable discontinuity* at  $a$  if  $\lim_{x \rightarrow a} f(x)$  exists but is not equal to  $f(a)$ .
2. A function  $f$  has a *jump discontinuity* at  $a$  if  $\lim_{x \rightarrow a^-} f(x)$  and  $\lim_{x \rightarrow a^+} f(x)$  both exist but are unequal.
3. A function  $f$  has a *infinite discontinuity* if  $f$  takes on arbitrarily large or small values near  $a$ . We'll talk about this more soon.
4. It's also possible for the one-sided limits to not exist, but this doesn't have a special name. We saw this with  $\sin(1/x)$ . In this class, I'll call a function like this *really bad*.

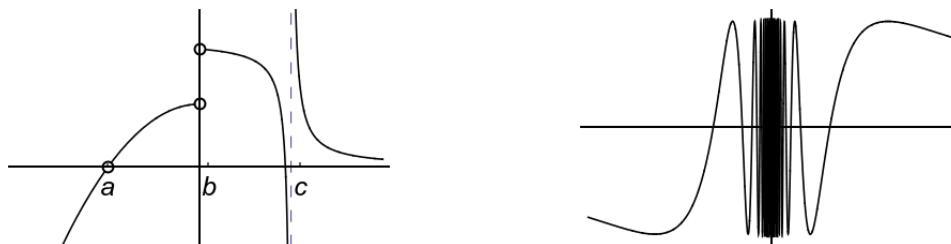


Figure 4.1: Left: a: removable discontinuity; b: jump discontinuity; c: infinite discontinuity. Right: really bad discontinuity

**Definition 4.2.** A function is *continuous from the right at a* if  $\lim_{x \rightarrow a^+} f(x) = f(a)$ .

A function is *continuous from the left at a* if  $\lim_{x \rightarrow a^-} f(x) = f(a)$ .

**Proposition 4.3.** A function is *continuous at a* if and only if it is *continuous from the left and from the right at a*.

**Example 4.4.** Recall the Heaviside function  $H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

This function is continuous from the right at 0, since  $\lim_{x \rightarrow 0^+} H(x) = 1 = H(0)$ . This function is not continuous from the left, since  $\lim_{x \rightarrow 0^-} H(x) = 0 \neq H(0)$ .

*Remark 4.5.* At a jump discontinuity, a function will often be continuous from one side but not the other. This is not necessarily the case, though: consider the function

$$f(x) = \begin{cases} 2 & x > 0 \\ 1 & x = 0 \\ 0 & x < 0 \end{cases}$$

Limits exist from the right and the left, but the function is not continuous from either side.

**Definition 4.6.** A function is *continuous on a set* if it is continuous at every number in that set.

A function is *continuous* if it is continuous at every number in its domain.

**Example 4.7.** The function  $f(x) = 1/x$  is continuous on the set  $\{x : x \neq 0\}$ . Since this is also the domain of  $f(x)$ , this function is continuous.

The Heaviside function is continuous on  $\{x : x \neq 0\}$ . However, it is not continuous, because 0 is in the domain.

**Proposition 4.8.** If  $f$  and  $g$  are continuous at  $a$  and  $c$  is a constant real number, then the following functions are all continuous at  $a$ :

1.  $f + g$
2.  $f - g$
3.  $cf$
4.  $f \cdot g$
5.  $\frac{f}{g}$  if  $g(a) \neq 0$ .

This in fact provides a proof of our earlier statement, that polynomials are continuous at all reals, and rational functions are continuous on their domains.

**Proposition 4.9.** *Root functions and trigonometric functions are continuous on their domains.*

**Example 4.10.** The function

$$f(x) = \frac{x^3 - 5x + 1}{(x - 1)(x - 2)(x - 3)}$$

is continuous exactly on its domain, which is  $\{x : x \neq 1, 2, 3\}$ .

**Proposition 4.11.** *If  $\lim_{x \rightarrow a} g(x) = b$  and  $f$  is continuous at  $b$ , then  $\lim_{x \rightarrow a} f(g(x)) = f(b)$ .*

*If  $g$  is continuous at  $a$  and  $f$  is continuous at  $g(a)$  then  $f \circ g$  is continuous at  $a$ .*

**Corollary 4.12.** *If  $f$  and  $g$  are continuous then so is  $f \circ g$ .*

**Example 4.13.** Where is  $\sqrt{1 + x^3}$  continuous?

Answer: Root functions are continuous on their domains.  $1 + x^3 \geq 0$  when  $x \geq -1$  so the function is continuous on its domain,  $[-1, +\infty)$ .

*Remark 4.14.* This is the big payoff for our discussion of continuous functions. Any function built up in a reasonable way out of rational functions, roots, and trigonometric functions (and a couple other types we will study later) will be continuous on its domain, and thus it is easy to take limits where these functions are defined.

Recall our goal was to figure out what values a function “should” have at points where it is not defined. If we have a function that is undefined for some real numbers, we’d often like to “extend” the function to one that is defined there.

**Definition 4.15.** We say that  $g$  is an *extension* of  $f$  if the domain of  $g$  contains the domain of  $f$ , and  $g(x) = f(x)$  whenever  $f(x)$  is defined.

*Remark 4.16.* We often wish to extend a function to one that is continuous at all reals. We can do this if and only if all discontinuities are removeable.

We have been implicitly discussing this idea since we defined limits the second week of class; now we can be more precise about what we’re doing.

**Example 4.17.** Can we extend  $x \sin(1/x)$  to a function continuous at all reals?

We can, but we have to be a little bit “cute”.  $x \sin(1/x)$  is continuous everywhere except at 0, where it is not defined. We have seen that  $\lim_{x \rightarrow 0} x \sin(1/x) = 0$ , so a continuous extension is

$$g(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

**Example 4.18.** The function  $f(x) = 1/x$  is continuous on its domain, but we cannot extend it to a function continuous at all reals, because the limit at 0 does not exist.

**Example 4.19.** Can we extend

$$f(x) = \frac{\sin(x)}{x}$$

to a function that is continuous on all reals?

Yes. The function is defined (and thus continuous) at all reals except 0.  $\lim_{x \rightarrow 0} f(x) = 1$ , so we can define

$$f_f(x) = \begin{cases} f(x) & x \neq 0 \\ 1 & x = 0 \end{cases}$$

which is the same as  $f$  on its domain, and is continuous on all reals.

**Example 4.20.** Let  $f(x) = \frac{x^2-1}{x-1}$ . Can we define a function  $g$  that agrees with  $f$  on its domain, and is continuous at all reals?

$f$  is continuous everywhere on its domain, and is undefined at  $x = 1$ . We can see that  $g(x) = x + 1$  will give the same value as  $f$  everywhere on  $f$ 's domain, and it is continuous since it is a polynomial.

*Poll Question 4.1.1.* When is

$$\frac{\tan(x) - \sqrt{x}}{(x-1)}$$

continuous?

$\tan$  and  $\sqrt{\phantom{x}}$  are both continuous functions, as are rational functions, so this will be continuous on its domain. Thus this is continuous on  $\{x : x \geq 0, x \neq 1, x \neq \pi(n + 1/2)\}$ .

*Poll Question 4.1.2.* Let  $f(x) = \frac{x^2-4x+3}{x-3}$ . Can we extend  $f$  to a function continuous at all reals?

Answer:  $f$  is continuous at all reals except  $x = 3$ . But the function  $g(x) = x - 1$  is the same everywhere except for 3, and is continuous at 3.

*Poll Question 4.1.3.* Let

$$g(x) = \begin{cases} x^2 + 1 & x > 2 \\ 9 - 2x & x < 2 \end{cases}$$

Can we extend this to a continuous function on all reals?

Answer:  $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} 9 - 2x = 5$ , and  $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} x^2 + 1 = 5$ , so the limit at 2 exists. Thus we can extend  $g$  to

$$g_f(x) = \begin{cases} x^2 + 1 & x \geq 2 \\ 9 - 2x & x \leq 2 \end{cases}$$

which is continuous at all reals.

## 4.2 The Intermediate Value Theorem

People will often describe the property of being “continuous” heuristically, as a function whose graph is one connected curve. It’s not true that every graph like that is a continuous function; but it is true that the graph of a continuous function will always be connected. That’s the content of the following theorem:

**Theorem 4.21** (Intermediate Value Theorem). *Suppose  $f$  is continuous (and defined!) on the closed interval  $[a, b]$  and  $y$  is any number between  $f(a)$  and  $f(b)$ . Then there is a  $c$  in  $(a, b)$  with  $f(c) = y$ .*

**Example 4.22.** Suppose  $f(x)$  is a continuous function with  $f(0) = 3$ ,  $f(2) = 7$ . Then by the Intermediate Value Theorem there is a number  $c$  in  $(0, 2)$  with  $f(c) = 5$ .

**Example 4.23.** Let  $g(x) = x^3 - x + 1$ . Use the Intermediate Value Theorem to show that there is a number  $c$  such that  $g(c) = 4$ .

To use the intermediate value theorem, we need to check that our function is continuous, and then find one input whose output is less than 4, and another whose output is greater than 4.  $g$  is a polynomial and thus continuous. Testing a few values, we see  $g(0) = 1$ ,  $g(1) = 1$ ,  $g(2) = 7$ . Since  $g(1) = 1 < 4 < 7 = g(2)$ , by the Intermediate Value Theorem there is a  $c$  in  $(1, 2)$  with  $g(c) = 4$ .

**Example 4.24.** Show that there is a  $\theta$  in  $(0, \pi/2)$  such that  $\sin(\theta) = 1/3$ .

We know that  $\sin$  is a continuous function, and that  $\sin(0) = 0$  and  $\sin(\pi/2) = 1$ . Since  $0 < 1/3 < 1$ , by the Intermediate Value Theorem there is a  $\theta$  in  $(0, \pi/2)$  such that  $\sin(\theta) = 1/3$ .

**Definition 4.25.** We say a function  $f$  has a *root* at  $c$ , or that  $c$  is a root of  $f$ , if  $f(c) = 0$ .

We often use the Intermediate Value Theorem to show that a continuous function must have a root.

**Example 4.26.** Show that  $f(x) = x - \sqrt{x} - 1$  has a root.

$f$  is continuous since it is composed of polynomials and roots. Plugging in values, we see that  $f(0) = -1, f(1) = -1, f(2) = 1 - \sqrt{2}, f(3) = 2 - \sqrt{3}, f(4) = 1$ . We see that  $f(0) = -1 < 0 < 1 = f(4)$ , so by the Intermediate Value Theorem there is a  $c$  between 0 and 4 such that  $f(c) = 0$ .

*Poll Question 4.2.1.* Does  $x^2 + 2$  have a root?

Does  $x^2 - 2$  have a root?

Why are these two situations different?

*Poll Question 4.2.2.* Does  $\cos x = x$  anywhere between 0 and 1? Between 1 and 2?

**Example 4.27.** Use the Intermediate Value Theorem to show that  $f(x) = \tan(x) + \sin(x) - 1$  has a root between 0 and  $\pi/4$ .

$f$  is continuous since it is made of trig functions (we need to check that every point between 0 and  $\pi/4$  is in the domain, but this is true).  $f(0) = 0 + 0 - 1 = -1$  and  $f(\pi/4) = 1 + \sqrt{2}/2 - 1 = \sqrt{2}/2$ . Since  $-1 < 0 < \sqrt{2}/2$ , by the Intermediate Value Theorem  $f$  has a root in  $(0, \pi/4)$ .

**Example 4.28.** Does  $g(x) = \tan(x)$  have a root between  $\pi/4$  and  $3\pi/4$ ?

$g$  is continuous where defined, and  $g(\pi/4) = 1, g(3\pi/4) = -1$ . However,  $g$  is not defined on the closed interval  $[\pi/4, 3\pi/4]$  since  $\tan(\pi/2)$  is not defined. Thus the Intermediate Value Theorem does not apply. Graphing the function, we can see it has no roots on this interval.

**Example 4.29.** Use the Intermediate Value Theorem to show that  $h(x) = x^4 - 3$  has two *distinct* roots.

As usual, we begin by plugging in some values. We see that  $h(-2) = 13, h(-1) = -2, h(0) = -3, h(1) = -2, h(2) = 13$ . Thus since  $h(-2) = 13 > 0 > -3 = h(0)$ , by the Intermediate Value Theorem  $h$  has a root between  $-2$  and 0. Similarly, since  $h(0) = -3 < 0 < 13 = h(2)$ , by the Intermediate Value Theorem  $h$  has a root between 0 and 2.

Since no number can simultaneously be in  $(-2, 0)$  and  $(0, 2)$ , these two roots cannot be the same. Thus  $h$  has two distinct roots.

**Example 4.30.** Use the Intermediate Value Theorem to show that  $f(x) = x^3 - 3x + 1$  has *three* distinct roots.

This function is a polynomial and thus continuous everywhere. Plugging in values gives us  $f(-2) = -1, f(-1) = 1, f(0) = 1, f(1) = -1, f(2) = 3$ . Thus by the intermediate value theorem there is a root between  $-2$  and  $-1$ , a root between 0 and 1, and a root between 1 and 2.

**Example 4.31.** Prove that every odd-degree polynomial has a root.

This is a bit trickier, but extremely useful! We know that all polynomials are continuous. We will see in the next section that every odd-degree polynomial will eventually take on very large values in one direction, and very negative values in the other direction. In particular, every odd-degree polynomial outputs a positive output for some input, and a negative output for some input. Thus by the Intermediate Value Theorem, it must have 0 as an output for some input.