5 Derivatives

5.1 The idea of a derivative

Speed is defined to be distance covered divided by time spent; that is, $v = \frac{\Delta x}{\Delta t}$. Unfortunately, this doesn't let us talk about your speed "at" some given time, since at a fixed point in time Δt is zero.

But this is exactly what we said limits were for—studying functions where they don't seem to be defined. We can talk about your "speed at time t" by taking the limit of your speed as Δt approaches zero.

If your position at a time t is given by a function f, then the distance you cover betwee between time t_0 and time t_1 is $f(t_1) - f(t_0)$, and the time it takes is $t_1 - t_0$. Thus the average speed over this interval is given by

$$\frac{f(t_1) - f(t_0)}{t_1 - t_0}.$$

(This formula should look familiar at this point).

Thus we can define your *instantaneous speed* or *speed at time t* to be

$$\lim_{t_1 \to t_0} \frac{f(t_1) - f(t_0)}{t_1 - t_0} = \lim_{h \to t_0} \frac{f(t_0 + h) - f(t_0)}{h}.$$

For example, on Earth dropped objects fall about $f(t) = 5t^2$ meters after t seconds. The average speed between time $t = 1$ and time $t = 2$ is

$$\frac{f(2) - f(1)}{2 - 1} = \frac{20 - 5}{1} = 15$$

meters per second, and the average speed between time $t = 0$ and time $t = 1$ is

$$\frac{f(1) - f(0)}{1 - 0} = 5$$

meters per second. We can generalize further, and ask for the average speed between time $t = 1$ and time $t = 1 + h$, for some interval h. Then we have

$$\frac{f(1 + h) - f(1)}{1 + h - 1} = \frac{5(1 + h)^2 - 5}{h} = 5\frac{h^2 + 2h + 1 - 1}{h} = 5(h + 2).$$

Thus the average speed over the interval $[1, 1 + h]$ is $5(h + 2)$. Plugging in earlier values: for $[1, 2]$ we take $h = 1$ and the speed is $5 \cdot 3 = 15$, and for $[0, 1]$ we take $h = -1$ and the speed is $5 \cdot 1 = 5$.

To find the speed "at" time 1, we calculate

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}5(h+2)=10.$$

Thus we say the speed "at" $t=1$ is 10 meters per second.

In general, if an object has position $f(t)$ at time t, we can find its average speed over some time interval by the formula

$$\frac{\nabla f}{\nabla t}=\frac{f(t_2)-f(t_1)}{t_2-t_1},$$

and we can find the speed at a time t_0 by taking the limit as this time interval gets very small:

$$\lim_{t\to t_0}\frac{f(t)-f(t_0)}{t-t_0}=\lim_{h\to 0}\frac{f(t_0+h)-f(t_0)}{h}.$$

Note these two expressions are completely equivalent—if we think of setting $t=t_0+h$ they are in fact exactly the same.

Example 5.1. If your position is given by $f(t)=t^2+t$, then what is your average speed over $[2,3]$? What is your average speed between time 2 and time $2+h$? What is your speed at time $t=2$?

The average speed over $[2,3]$ is

$$\frac{f(3)-f(2)}{3-2}=\frac{12-6}{1}=6.$$

The average speed between 2 and $2+h$ is

$$\frac{f(2+h)-f(2)}{h}=\frac{(2+h)^2+(2+h)-6}{h}=\frac{5h+h^2}{h}=5+h.$$

The speed at time $t=2$ is

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{(2+h)^2+(2+h)-6}{h}=\lim_{h\to 0}\frac{5h+h^2}{h}=\lim_{h\to 0}5+h=5.$$

5.2 The derivative defined

The preceeding discussion motivates the following definition:

Definition 5.2. Let f be a function defined near and at a point a. We say the *derivative* of f at a is

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}.$$

We will often write $f'(a)$ or $\frac{df}{dx}(a)$ for the derivative of f at a. The first is sometimes called "Newtonian notation," while the second is "Leibniz notation."

Remark 5.3. Note that we need *two* pieces of information here. You hand me a function f *and* a point a, and I tell you the derivative of f at a. We'll adopt different perspectives from time to time later on in the course.

We can interpret the derivative in a number of ways. If f represents the position of an object, $f'(a)$ represents the speed at the time a. $f'(a)$ is the slope of the tangent line to the graph of f at a. More generally, $f'(a)$ describes "how fast" the output of f changes when the input changes a little bit.

Example 5.4.

1. Let $f(x) = x^2 + 1$. Then
$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{(2+h)^2 + 1 - 2^2 - 1}{h} = \lim_{h \to 0} \frac{4h + h^2}{h} = 4,$$
and more generally, for any number a we have
$$f'(a) = \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h \to 0} \frac{2ah + h^2}{h} = 2a.$$

2. Let $f(x) = x^3$, and let's find the derivative at a point a. Then
$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^3 - a^3}{x - a} \\ &= \lim_{x \to a} \frac{(x-a)(x^2 + ax + a^2)}{x - a} = \lim_{x \to a} x^2 + ax + a^2 = 3a^2. \end{aligned}$$
Notice that it wasn't obvious that we could factor $x^3 - a^3$ this way. We could notice this by noticing that plugging in a gives us zero; in general, if plugging a into a polynomial gives zero, we can always factor out a $(x - a)$ term. In this case, though, it might have been easier to just start with the limit as $h \to 0$, in which case the problem would have essentially solved itself.

3. Let $f(x) = \sqrt{x}$. Then given a number a, we have
$$f'(a) = \lim_{h \to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} = \lim_{h \to 0} \frac{(a+h) - a}{h(\sqrt{a+h} + \sqrt{a})} = \lim_{h \to 0} \frac{1}{\sqrt{a+h} + \sqrt{a}} = \frac{1}{2\sqrt{a}}$$
Note that f is defined at 0, and we have $f(0) = 0$. But by this computation we have $f'(0) = \frac{1}{2 \cdot 0}$ which is undefined. This isn't an artifact of the way we computed it; the limit in fact does not exist. Further, this isn't just becasue 0 is on the edge of the domain of f, as we shall see:

4. Let $g(x) = \sqrt[3]{x}$. Then we can compute $g'(0)$ and we get
$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{\sqrt[3]{h}}{h} = \lim_{h \to 0} \frac{1}{\sqrt[3]{h^2}} = +\infty.$$

The cube root function g has no defined derivative at 0, even though the function is defined there. This brings us to a discussion of ways for a function to fail to be differentiable at a point. (There's always the catchall category of "the limit just doesn't exist," which we won't really discuss because there's not much to say about it).

Example 5.5.

1. Our first example of $g(x) = \sqrt[3]{x}$ is not differentiable at 0, and the limit
$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = +\infty.$$
Graphically, the line tangent to g at 0 is completely vertical; the function is "increasing infinitely fast" at 0.

2. Any function that is not continuous at a point cannot be differentiable at that point. In particular, if f is differentiable at a, then
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
converges. But the bottom goes to zero, so the top must also go to zero, and we have
$$\lim_{x \to a} f(x) = f(a),$$
which is precisely waht it means to be continuous.

   Conceptually, if the function isn't continuous, it isn't changing smoothly and so doesn't have a "speed" of change. Graphically, a function that has a disconnect in it doesn't have a clear tangent line.

   An example here is the Heaviside function $H(x)$. We have
$$\lim_{h \to 0^+} \frac{H(h) - H(0)}{h} = \lim_{h \to 0^+} \frac{0}{h} = 0$$
but
$$\lim_{h \to 0^-} \frac{H(h) - H(0)}{h} = \lim_{h \to 0^-} \frac{-1}{h} = +\infty.$$
Since the one-sided limits aren't equal, the limit does not exist.

3. Any function with a sharp corner at a point doesn't have a well-defined rate of change at that point; the change is instantaneous. For instance, if we let $a(x) = |x|$ be the absolute value function, then
$$a'(x) = \lim_{h \to 0} \frac{a(x+h) - a(x)}{h}.$$

To study piecewise functions we usually break them up and study each piece separately. If $x > 0$, then $a(x) = x$ and $a(x+h) = x+h$ for small h. We have

$$a'(x) = \lim_{h \to 0} \frac{x+h-x}{h} = \lim_{h \to 0} 1 = 1.$$

Conversely, if $x < 0$ then $a(x) = -x$ and $a(x+h) = -x-h$, and

$$a'(x) = \lim_{h \to 0} \frac{-x-h+x}{h} = \lim_{h \to 0} -1 = -1.$$

But if $x = 0$ then the left and right limits don't agree again: the right limit is 1 and the left limit is -1, so the limit does not exist. Thus we have

$$a'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ \text{undefined} & x = 0. \end{cases}$$

4. Sometimes a function has a "cusp" at a point. This is a point where the tangent line is vertical, but depending on the side from which you approach, you can get a tangent line that goes up incredibly fast or one that goes down incredibly fast.

   Consider the funtion $f(x) = \sqrt[3]{x^2}$. We have

   $$f'(0) = \lim_{h \to 0} \frac{\sqrt[3]{h^2} - \sqrt[3]{0}}{h} = \lim_{h \to 0} \frac{h^{2/3}}{h} = \lim_{h \to 0} \frac{1}{\sqrt[3]{h}} = \pm\infty.$$

   This is different from the $\sqrt[3]{x}$ example because the limit is $\pm\infty$ rather than just $+\infty$.

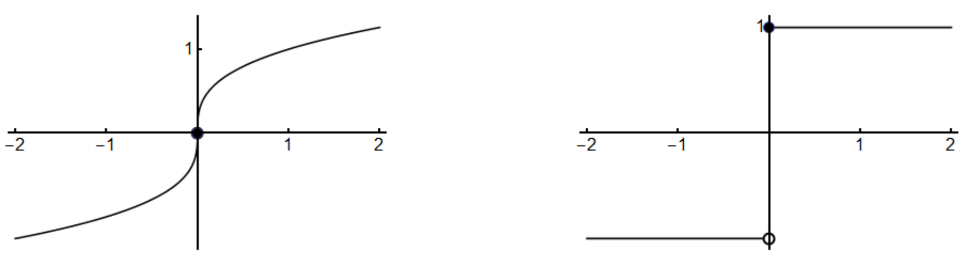

Figure 5.1: A vertical tangent line and a discontinuous function

Poll Question 5.2.1. Let $f(x) = \sqrt{x^2 - 4}$. What is $f'(x)$? Where is f differentiable?

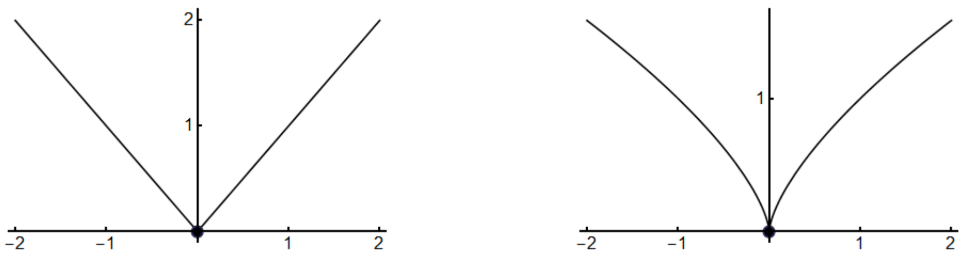

Figure 5.2: A corner and a cusp

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{(x+h)^2-4} - \sqrt{x^2-4}}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - 4 - (x^2-4)}{h(\sqrt{(x+h)^2-4} + \sqrt{x^2-4})} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h(\sqrt{(x+h)^2-4} + \sqrt{x^2-4})} \\
&= \lim_{h\to 0} \frac{2x + h}{(\sqrt{(x+h)^2-4} + \sqrt{x^2-4})} \\
&= \frac{2x}{2\sqrt{x^2+4}} = \frac{x}{\sqrt{x^2-4}}.
\end{aligned}$$

Thus we see that f is differentiable on $(-\infty, -2) \cup (2, +\infty)$.

5.3 The derivative as a function

Our computation of the derivative of $|\cdot|$, and of several other functions, looks a lot like a function itself. Taking the derivative of a function f in fact gives us a new function f': the rule of this function is that given a number a, we compute the derivative of f at a and return that as our output. Thus f' is a function and we can study it the way we did earlier functions.

Definition 5.6. The *derivative of a function* f is the function

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Example 5.7.
1. If $f(x) = x^2 + 1$, we computed that $f'(x) = 2x$. The domain of f is all reals, and so is the domain of $f'(x)$.

2. If $g(x) = \sqrt{x}$ then $g'(x) = \frac{1}{2\sqrt{x}}$. The domain of g is all reals ≥ 0, and the domain of g' is all reals > 0.

3. We saw above that if $a(x) = |x|$, then

$$a'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ \text{undefined} & x = 0 \end{cases} = \frac{|x|}{x}.$$

The domain of a is all reals and the domain of a' is all reals except 0.

Further, since f' is a function we can ask about the derivative of f' at a point a.

Definition 5.8. Let f be a function which is differentiable at and near a point a. The *second derivative of f at a* is the derivative of the function $f'(x)$ at a, which is

$$f''(a) = \lim_{h \to 0} \frac{f'(a+h) - f'(x)}{h} = \frac{d^2 f}{dx^2}(a).$$

This is again a limit and may or may not exist.

Remark 5.9. The Leibniz notation for a second derivative is $\frac{d^2 f}{dx^2}$ and not $\frac{df^2}{dx^2}$. Conceptually, you can think of $\frac{d}{dx}$ as a function whose input is the function f and whose output is the derivative function f'. The second derivative results from applying this function twice.

Poll Question 5.3.1. What is the second derivative of $f(x) = x^3$ at $a = 2$?

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{3x^2 h + 3h^2 + h^3}{h} = \lim_{h \to 0} 3x^2 + 3h + h^2 = 3x^2.$$

$$\begin{aligned} f''(2) &= \lim_{h \to 0} \frac{f'(2+h) - f'(2)}{h} = \lim_{h \to 0} \frac{3(2+h)^2 - 3 \cdot 2^2}{h} = \lim_{h \to 0} \frac{3(4 + 4h + h^2) - 12}{h} \\ &= \lim_{h \to 0} \frac{12h + 3h^2}{h} = \lim_{h \to 0} 12 + 3h = 12. \end{aligned}$$