Lab 8 Thursday April 12

Exponential Growth

Discrete Compound Growth

Often we have something that grows at a percentage rate. Say a population of bacteria is growing at a rate of 50% per hour. If it starts at 1000, then after an hour there will be 1500. After two hours there will be, not 2000, but 2250. We see that after t hours the population will be $1000 \cdot (1.5)^t$. We can write this formula as

$$P(t) = P(0) \cdot (1 + r)^t. \tag{1}$$

Interest payments in particular have an interesting extra issue. In the example above the interest was *compounded* once per year–meaning that each year a payment was made, taking into account the amount of money after the previous year. But sometimes payments were compounded more often. Rather than making a 4% payment every year, borrowers would make a 1% payment each of four times per year. In this case we can generalize our formula: if your payment is compounded n times per year, then

$$P(t) = P(0) \cdot (1 + r/n)^{tn}. \tag{2}$$

Continuous compound growth

The mathematician Jacob Bernoulli in the late 1600s was interested in the idea of *continuously compounded interest*, which is the limit as the rate of compounding goes to infinity. This also makes more sense in many natural situations: most growth doesn't happen in distinct instantaneous spikes, but happens continuously over time. So we might expect this to show up in nature.

Mathematically, Bernoulli asked for the limit $\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n$. You might recognize this from the definition of e; in fact, we have

$$\lim_{n\to+\infty} (1 + 1/n)^n = e$$

and thus Bernoulli's limit was

$$\lim_{n\to+\infty} (1 + r/n)^n = \lim_{n/r\to+\infty} \left(1 + \frac{1}{n/r}\right)^{rn/r} = \left(\lim_{n/r\to+\infty} \left(1 + \frac{1}{n/r}\right)^{n/r}\right)^r = e^r.$$

Thus when something is epxeriencing continuous compound growth at a rate of r, we have

$$P(t) = P(0) \cdot e^{rt}. \tag{3}$$

We can find this same equation a totally different way, through differential equations. In these situations of compound growth, the derivative of the population is not a constant–as the population gets bigger, it grows faster. (An increase of 50% is bigger when you start with 1000 than when you start with 10). So the derivative $P'(t)$ is definitely not constant.

However, the "growth rate" as a proportion of the total population is a constant. We can express this as

$$P'(t) = rP(t) \tag{4}$$

for some rate r, which we call the *relative growth rate*.

But we can see that the function of equation (3) satisfies this differential equation! For

$$P'(t) = P(0) \cdot e^{rt} \cdot r = rP(t).$$

In Calculus 2 you will see that this solution is unique up to a constant factor (which simply represents the initial population). Thus equation (3) is the only one we could possibly have if the relative growth rate is a constant.

Thus whenever we know we have a constant relative growth rate, we know it must be described by equation (3). We then only need to find two points to determine $P(0)$ and r, in order to find a complete equation.

Exercise Solutions

Discrete Compound Growth

1. Suppose a population of animals grows discretely by 3% a year. If we start with 1000 animals, how many will we have after five years?

   Solution:
   $$1000 \cdot (1.03)^5 \approx 1159.$$

2. (a) Many bacterial populations, in the absence of resource restrictions, will double their populations every thirty minutes. If you start with 10,000 bacteria, how many are there after a day?

   Solution:
   $$10000 * 2^{48} = 2814749767106560000 \approx 2.8 \cdot 10^{18}$$

   (b) If the average bacterium is 10^{-12} grams, what is the mass of this bacterial growth? (Note: this should tell you how important those resource constraints are).

   Solution: Approximately $2,814,749$ grams, or about three tons. We should all feel grateful that our bodies are not replaced by three tons of bacteria after every day.

3. (a) Suppose you invest \$100 at 4% per year, such that you have \$104 after one year. How much money will you have after five years? After ten? After 20?

   Solution:
   $$100 \cdot (1.04)^5 \approx 122 \qquad 100 \cdot (1.04)^{10} \approx 148 \qquad 100 \cdot (1.04)^{20} \approx 219.$$

   (b) If your 4% interest is compounded twice per year, then you get a payment of 2% of your **current** account balance on July 1 and on January 1. The first payment will increase your \$100 investment to \$102; after the second payment, how much will you have?

   Solution: $100 * 1.02 * 1.02 = 104.04$.

   (c) What if it is compounded four times per year?

   Solution: $100 \cdot 1.01 \cdot 1.01 \cdot 1.01 \cdot 1.01 \approx 104.06$.

   (d) If you invest \$100 at a rate of 4%, how much will you have after ten years if you compound annually? What if you compound quarterly? Four hundred times per year? Which would you prefer?

   Solution:
   $$\begin{aligned} \text{annually: } 100(1.04)^{10} &= 148.024 \\ \text{quarterly: } 100(1.01)^{40} &= 148.886 \\ 400\text{: } 100(1.0001)^{4000} &= 149.179 \end{aligned}$$

   We see we get the most money if we compound the most often, at four hundred times per year. (If we compounded four thousand times per year, of course it would get even better. And it would be best if we compounded continuously).

Continuous Compound Growth

1. **Continuously Compounded Interest**

   (a) If you invest \$100 at a rate of 4%, how much will you have after ten years if you compound continuously? Is this better or worse than the earlier options?

   Solution:
   $$100e^{.04 \cdot 10} \approx 149.182$$
   This is marginally better than compounding four hundred times per year. But not by much!

   (b) If you invest \$1000 at a rate of 3% compounded continuously, write an equation for how much money you will have after t years. (This equation should involve the number e). How much money will you have after five years? After ten?

   Solution: We get $M(t) = 1000e^{.03t}$. Thus $M(5) \approx 1162$ and $M(10) \approx 1350$.

2. **Population modelling**

   (a) In real life, we often want to find solutions to differential equations in order to model something we've observed. To do this we need an "initial condition"–some particular data point. If a population is growing at a monthly rate of 10% and has 100 members at the start of the process, what is the equation for $P(t)$?

   Solution:
   $$P(t) = 100e^(.1t)$$

   (b) What if the population is growing at a monthly rate of 5% and has 116 members after three months (at $t = 3$)? What is the equation in this case?

   Solution: We have $P(t) = Ce^{.05t}$ and $P(3) = 116$. Thus $116 = Ce^{.05 \cdot 3}$ and thus $C = 116/e^{.15} \approx 100$.

   (c) What if the population is growing at a yearly rate of 7% and has 71 members after five years?

   Solution: We have $P(t) = Ce^{.07t}$ and $P(5) = 71$. Thus $71 = Ce^{.35}$ and $C = 71/e^{.35} \approx 50$.

3. **Human Global Population Modelling**

   We can use these tools to model the global growth of **human** population. If we assume population growth is exponential, we generally get pretty reasonable numbers.

   (a) The population of earth was 3 billion people in 1960, and 4 billion people in 1975. Write an equation giving the population of the earth as a function of time.

   Solution: It's easiest to take $t = 0$ to be the year 1960, and measure in billions. We have $P(t) = Ce^{rt}$, and we know $P(0) = 3$ and $P(15) = 4$. Thus $3 = Ce^0 = C$, and we have $4 = 3e^{15r}$. This gives us $\ln(4/3) = 15r$ adn thus $r = \ln(4/3)/15$, and therefore
   $$P(t) = 3e^{\ln(4/3)t/15}.$$
   If you want a decimal approximation, we can compute that $\ln(4/3)/15 \approx .019$.

(b) What do you estimate the population to be in 2020?

Solution: $P(60) = 3e^{\ln(4/3)\cdot 60/15} = 3e^{4\ln(4/3)} = 3(4/3)^4 = \frac{2^8}{27} \approx 9.48$. Thus we estimate about 9.48 billion people in 2020.

(c) In what year would you expect population to hit 12 billion?

Solution: We solve the equation $12 = 3e^{\ln(4/3)t/15}$ and thus $\ln(4) = \ln(4/3)t/15$, hence $t = 15\ln(4)/\ln(4/3) \approx 72$. Thus we expect the population to hit 12 billion in 2032.

(d) The actual population in 2020 is estimated to be 7.7 billion. What does this tell you about your model?

Solution: Our model is an overestimate. This suggests that the rate of population growth has been and is decreasing. (Indeed, we get much better results with a model called *logistic growth*, which takes into account the idea that growth rates might decline as populations get closer to carrying capacity).

4. **Radioactive Decay**

Another standard example of a natural phenomenon with this behavior is radioactive decay. Radioactive substances decay randomly: each atom has a random chance of decaying at any given time. Thus the number of atoms that decay is proportional to the total number of atoms. If $M(t)$ is the mass of radioactive substance, then $M'(t) = kM(t)$. The speed of radioactive decay is often described by the "half-life," which is the amount of time it takes for half of a substance to decay.

(a) The half-life of Radium-226 is 1590 years. How much of a 100g mass will be left after 1000 years? (Note that here you need to figure out both k and $M(0)$).

Solution: First we figure out the decay rate. We have $M(t) = Ce^{kt}$ and $M(1590) = \frac{1}{2}M(0)$, and thus we have $1/2 = e^{1590k}$. Taking logs and dividing gives us $k = \ln(1/2)/1590 = -\ln(2)/1590$. (You may or may not recall that $\ln(2) \approx .7$).

Thus $M(t) = 100e^{-\ln(2)t/1590}$. In particular, we have $M(1000) = 100e^{-\ln(2)\cdot 1000/1590} = 100 \cdot (1/2)^{1000/1590} \approx 65$. So about 65 grams will be left. This seems reasonable since it will take another 600 years to get down to 50 grams.

(b) How many years will it take to have 30 grams left?

Solution: We already know that $M(t) = 100e^{-\ln(2)t/1590}$, so we solve $30 = 100e^{-\ln(2)t/1590}$. This gives $\ln(.3) = -\ln(2)t/1590$ or $t = -1590\ln(.3)/\ln(2) \approx 2761$. So it takes about 2761 years for the sample to reach 30 grams; this is again reasonable since after about 3200 it will have 25 grams.

(c) The half-life of carbon-14 is 5730 years. If an object had 100g of carbon-14 originally, and has 60g now, how old is it?

Solution: As before, we see that $1/2 = e^{5730k}$ and thus $k = -\ln(2)/5730$. Then we have $C(t) = 100e^{-\ln(2)t/5730}$. Now we solve $60 = 100e^{-\ln(2)t/5730}$, which gives $t = -5730\ln(.6)/\ln(2) \approx 4223$. Thus it will take about 4223 years to reach 60 grams; again, this is a little less than the half-life, which makes sense.