

Lab 10 Solutions

Thursday April 19

Newton's Method

1. (a) Starting with $x_1 = 1$, use $f[x_] := x^4 - 2$ to estimate $\sqrt[4]{2}$ to four decimal places.

We have $f'(x) = 4x^3$, so compute

$$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-1}{4} = 5/4$$

$$x_3 = 5/4 - \frac{f(5/4)}{f'(5/4)} = \frac{5}{4} - \frac{(625/256) - (512/256)}{125/16} = \frac{5}{4} - \frac{113/256}{125/16} = \frac{5}{4} - \frac{113}{125 \cdot 16} = \frac{2387}{2000}$$

Switching to a computer,

$$x_4 = \frac{2387}{2000} - \frac{f(2387/2000)}{f'(2387/2000)} \approx 1.18923$$

$$x_5 = 1.18923 - \frac{f(1.18923)}{f'(1.18923)} \approx 1.18921.$$

- (b) Plot the tangent line corresponding to each step you executed in part (a).
 (c) Repeat part (a) starting with $x_1 = -1$, and again with $x_1 = 0$.
 You should get -1.18921 in the first case, and in the second case you get a divide-by-zero error (literally—a “singular Jacobian” means a derivative equal to zero).
 (d) In Mathematica, run the commands `FindRoot[x^4 == 2, {x, 1}]`, `FindRoot[x^4 == 2, {x, -1}]`, and `FindRoot[x^4 == 2, {x, 0}]`.
2. (a) Plot a graph of both `Cos[x]` and `x` with the command `Plot[{Cos[x], x}, {x, -2Pi, 2Pi}]`. About where does it look like the two functions intersect?

- (b) Using your guess from part (a) as a starting point, use Newton's method to estimate a solution to $\cos(x) = x$ that is correct to six decimal places.

Note that we are not asking for a root of $\cos(x)$! We want to know when $\cos(x) = x$, that is, when $\cos(x) - x = 0$. So we want a root of $f(x) = \cos(x) - x$.

Starting with a guess of $x_1 = \pi/4$ (other guesses, such as 1, are also reasonable), we get

$$x_2 = \pi/4 - \frac{f(\pi/4)}{f'(\pi/4)} = \pi/4 - \frac{\sqrt{2}/2 - \pi/4}{-\sqrt{2}/2 - 1} \approx .739536$$

$$x_3 = .739536 - \frac{f(.739536)}{f'(.739536)} = .739085$$

- (c) Run the command `FindRoot[Cos[x]==x, {x,a}]`, where a is your guess from part (a).

3. (a) Starting with $x_1 = 1$, estimate the root to $g[x_] = x^3 - x - 1$ to four decimal places. We have $g'(x) = 3x^2 - 1$, so compute

$$x_2 = 1 - \frac{-1}{2} = \frac{3}{2}$$

$$x_3 = \frac{3}{2} - \frac{27/8 - 3/2 - 1}{27/4 - 1} \approx 1.34783$$

$$x_4 = 1.34783 - \frac{g(1.34783)}{g'(1.34783)} \approx 1.3252$$

$$x_5 \approx 1.32472$$

- (b) Do the same, starting with $x_1 = .6$.

This should take much longer

- (c) Do the same, starting with $x_1 = .57$.

This should take nearly a hundred iterations if it converges at all.

- (d) Plot g from -2 to 2 . Why were (a), (b), and (c) so different? Try plotting some tangent lines.

There is a local minimum at about $.577$. $.6$ is close to that but on the same side as the (only) root, so it converges but slowly. $.57$ is on the “wrong” side and gets “trapped” there for a long time.

4. (a) Starting with $x_1 = 1$, use three iterations of Newton’s method to find a solution to $\text{CubeRoot}[x] == 0$. What happens?

Starting with 1 we have

$$x_2 = 1 - \frac{\sqrt[3]{1}}{(1/3)1^{-2/3}} = 1 - 3 = -2$$

$$x_3 = -2 - \frac{\sqrt[3]{-2}}{1/3(-2)^{-2/3}} = -2 + 6 = 4$$

$$x_4 = -8$$

and so on. The method never converges, and in fact each new iteration gets us further from the root (which is zero) than the previous one did.

- (b) Plot a graph of $\text{CubeRoot}[x]$. Graphically, why did you get the result you did in part (a)? Plot the tangent lines that correspond to the approximations you calculated.

There is a vertical tangent line at the root.

5. Let $f(x) = x^5 + x^3 + x$.

- (a) Use Newton’s method to approximate $f^{-1}(2)$ —i.e., to approximate a solution for $f(x) = 2$.

We want a solution to $f(x) = 2$, so set $f_2(x) = x^5 + x^3 + x - 2$, and then we're looking for a root of f_2 . We have $f_2'(x) = 5x^4 + 3x^2 + 1$. Take $x_1 = 1$ since we know that $f(1) = 3$ is close to 2; then we have

$$\begin{aligned}x_2 &= 1 - \frac{f_2(1)}{f_2'(1)} = 1 - \frac{1}{9} = \frac{8}{9} \\x_3 &= \frac{8}{9} - \frac{f_2(8/9)}{f_2'(8/9)} = \frac{8}{9} - \frac{(8/9)^5 + (8/9)^3 + 8/9 - 2}{5(8/9)^4 + 3(8/9)^2 + 1} \approx .866376 \\x_4 &= .865583\end{aligned}$$

(b) Use The Implicit Function Theorem to approximate $(f^{-1})'(2)$.

We know from part (a) that $f^{-1}(2) \approx .865583$. So

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} \approx \frac{1}{f'(.865583)} \approx \frac{1}{6.05446} \approx .165168.$$

6. Let $g(x) = \sqrt{1 + x + x^2 + x^3}$.

(a) Use Newton's Method to approximate $g^{-1}(3)$.

Again, we want a root of $g(x) - 3$ so let $g_1(x) = \sqrt{1 + x + x^2 + x^3} - 3$. Then $g_1'(x) = \frac{1+2x+3x^2}{2\sqrt{1+x+x^2+x^3}}$.

We can take $x_1 = 0$. Then

$$\begin{aligned}x_2 &= 0 - \frac{g_1(0)}{g_1'(0)} = 0 - \frac{-2}{1/2} = 4 \\x_3 &= 4 - \frac{g_1(4)}{g_1'(4)} = 4 - \frac{\sqrt{85} - 3}{57/2\sqrt{85}} = 4 - \frac{2\sqrt{85}}{57} (\sqrt{85} - 3) \approx 1.988 \\x_4 &= x_3 - \frac{g_1(x_3)}{g_1'(x_3)} \approx 1.60.\end{aligned}$$

The true answer is approximately 1.578.

(b) Use the Implicit Function theorem to approximate $(g^{-1})'(1)$.

We have $g^{-1}(3) \approx 1.6$ from part (c), and thus

$$(g^{-1})'(1) = \frac{1}{g'(g^{-1}(1))} \frac{1}{g'(1.6)} \approx \frac{1}{1.95} \approx .51.$$

7. (a) Approximate $\sqrt[5]{20}$ to eight decimal places.

(b) Find four real roots of $x^6 - x^5 - 6x^4 - x^2 + x + 10$ to eight places.

(c) Show that $x^4 - 3x^3 + 5x^2 - 6$ has a root in (1,2) (hint: IVT), and approximate it to six decimal places.

(d) Approximate $\log 3$.