1 Multivariable Functions

1.1 Functions, twospace, and threespace

Recall that a *function* is a rule that takes an input and assigns a specific output. Note that a function always gives exactly one output, and always gives the same output for a given input.

Importantly, we haven't specified what types of outputs functions have—in the abstract, a function can take any type of input and give any type of output. We say that the set of possible inputs for a function is the *domain*.

In your previous calculus courses you have seen "single–variable" functions, which take in a single real number and output a single real number. In this course we will study functions of "multiple variables". That is, either their input, or their output, or both will consist of more than one real number.

Example 1.1.

- Altitude as a function of latitude and longitude
- Gas pressure as a function of volume and temperature
- Temperature as a function of latitude and longitude and altitude
- Volume of a box as a function of length and width and height
- Position as a functionof time
- Direction and distance to your destination as a function of where you are

Definition 1.2. We use $\mathbb{R}$ to denote the set of all real numbers.

We use $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ to denote the set of ordered pairs of real numbers. This is sometimes called the *real plane* or the *cartesian plane*.

We use $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ to denote the set of ordered triples of real numbers. This is sometimes called *real threespace*.

Remark 1.3. We can use $\mathbb{R}^n$ to refer to the set of ordered sets of n real numbers. This is an n-dimensional space. Everything we do in this course can be generalized to work in $\mathbb{R}^n$ rather than $\mathbb{R}^2$ or $\mathbb{R}^3$, but in this course we're going to confine ourselves to the two- and three-dimensional cases.

This is mainly because those cases are easier to draw. But it's also harder to talk about the higher-dimensional cases without linear algebra.

Geometrically, we think of $\mathbb{R}^2$ as representing a plane; the first real number gives the x-coordinate, or horizontal distance from the origin; the second gives the y-coordinate or vertical distance.

Similarly, $\mathbb{R}^3$ represents three-dimensional space, with the three numbers giving the distances along three perpendicular axes. We put these axes together according to the *right-hand rule*, which says that if we look **from** the positive side of the z-axis we see the xy-plane in the usual orientation.

Example 1.4.

We can compute the *distance* between two points using the pythagorean theorem. If (x, y) and (a, b) are two points in $\mathbb{R}^2$, then the distance between them is given by $\sqrt{(x-a)^2 + (y-b)^2}$. Similarly, if (x, y, z) and (a, b, c) are two points in $\mathbb{R}^3$, then the distance between them is given by $\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$.

Example 1.5. The distance between $(3, 5)$ and $(1, 7)$ is

$$\sqrt{(3-1)^2 + (5-7^2)} = \sqrt{4+4} = 2\sqrt{2}.$$

The distance betweent $(1, -1, 0)$ and $(3, 2, 1)$ is

$$\sqrt{(1-3)^2 + (-1-2)^2 + (0-1)^2} = \sqrt{4+9+1} = \sqrt{14}.$$

Example 1.6. Suppose we want to describe a sphere in threespace. The unit sphere is the set of all points of distance one from the origin. That is, it's the set of all points (x, y, z) such that $\sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = 1$. We can simplify this equation and see that the unit sphere is precisely the set of points that satisfy the equation $x^2 + y^2 + z^2 = 1$.

More generally, if we have a sphere of radius r centered at the point (a, b, c), then it is the set of all points of distance r from (a, b, c). So it's the set of all points such that $\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r$, or in other words $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$.

In this course we'll think about four or five basic categories of functions. This taxonomy is a bit artificial but it will still be useful to think about:

Type of function	Name	Example of use
$\mathbb{R} \to \mathbb{R}$	Single-variable functions	
$\mathbb{R}^2$ or $\mathbb{R}^3 \to \mathbb{R}$	Multivariable function	Temperature as a function of position
$\mathbb{R} \to \mathbb{R}^2$ or $\mathbb{R}^3$	Parametrization of a path	Position as a function of time
$\mathbb{R}^2 \to \mathbb{R}^3$	Parametrized surface	
$\mathbb{R}^2 \to \mathbb{R}^2$ or $\mathbb{R}^3 \to \mathbb{R}^3$	Vector field	Direction of a force as a function of position

You are already familiar with single-variable functions from your previous calculus courses. For the first part of this course we're going to focus on multivariable functions, that take in multiple variables but output a single variable. In the second half of the course we will look at functions that output more than one variable as well.

1.2 Graphing multivariable functions

To describe and understand single-variable functions, we would draw a graph, with one dimension representing the input and one dimension representing the output. We would like to do the same thing for multivariable functions, but the situation is a bit more difficult because it's much harder to draw three-dimensional pictures. (And all but impossible to draw four- or six-dimensional pictures).

1.2.1 Graphing functions of two variables as surfaces

Recall that when we graphed a single-variable function f, we took all the points (x, y) such that $y = f(x)$. Similarly, we can define:

Definition 1.7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables. Then the *graph* of f is the set $\{(x, y, z) : z = f(x, y)\}$ of all points with $z = f(x, y)$.

The graph of a two-variable function will generally look like a curved two-dimensional surface in three-dimensional space.

A graph of a two-variable function will still have to pass the vertical line test: a vertical line given by $x = a, y = b$ will intersect the surface in at most one point. This is because a given (x, y) input has only one output.

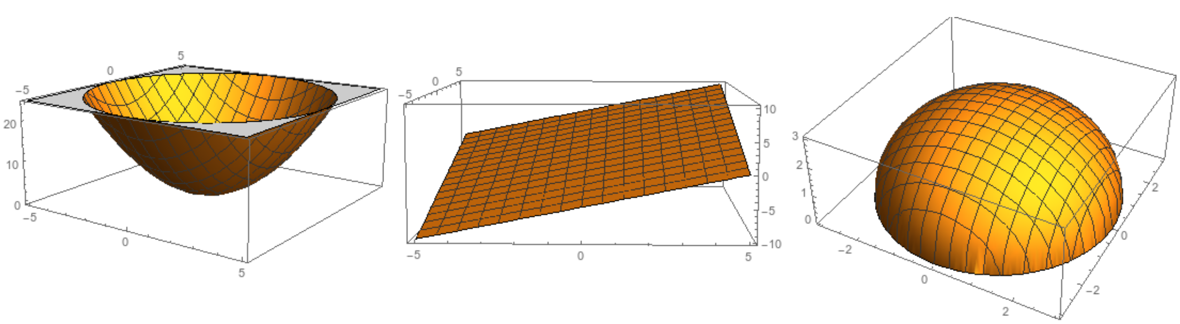


Figure 1.1: Graphs of the functions $x^2 + y^2$, $x + y$, and $\sqrt{9 - x^2 - y^2}$

1.2.2 Transformations of two-variable functions

If you already know the graph of one function, you can often figure out what the graphs of related functions must look like.

- The graph of $f(x, y) + c$ is the graph of $f(x, y)$ shifted up (along the z-axis) by c units.
- The graph of $f(x - a, y - b)$ is the same as the graph of $f(x, y)$ but shifted a units along the x axis and b units along the y axis. You can think of this as moving the center of the graph from $(0, 0)$ to (a, b).
- The graph of $f(-x, y)$ is the graph of $f(x, y)$ reflected across the yz plane, inverting the x axis.
- The graph of $f(x, -y)$ is the graph of $f(x, y)$ reflected across the xz plane, inverting the y axis.
- The graph of $-f(x, y)$ is the graph of $f(x, y)$ reflected across the xy plane, inverting the z axis and drawing the graph "upside down".

Example 1.8. Let's consider the function $f(x, y) = x^2 + y^2$ that we saw in figure 1.1. Then we can look at the following ways of shifting the function in figure 1.2:

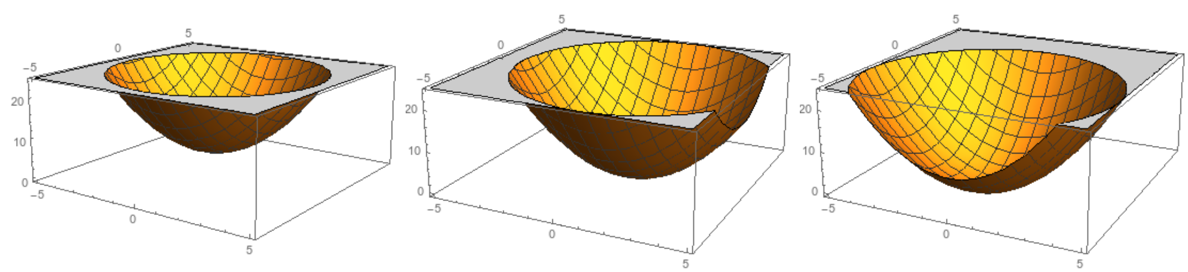


Figure 1.2: The graphs of $f(x, y) + 5$, $f(x - 1, y)$, and $f(x, y + 2)$

Similarly, we can take the function $g(x, y) = x + 5y$ and look at the following graphs in figure 1.3:

1.2.3 Graphing two-variable functions with cross-sections

We still don't have a good way to figure out what the graph of a two-variable function looks like if we don't already know. But the last section gives us an idea: look at each variable individually.

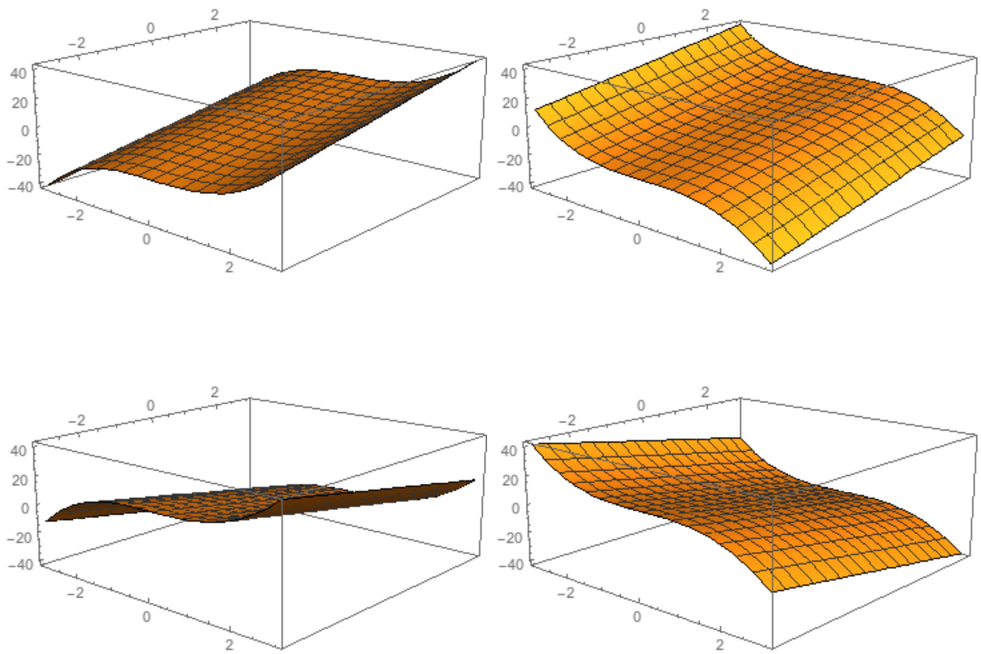


Figure 1.3: The graphs of $g(x, y)$, $g(-x, y)$, $g(x, -y)$, and $-g(x, y)$

Definition 1.9. If $f(x, y)$ is a function of two variables, then we can get a function of one-variable by fixing $x = c$ and considering the function $f(c, y)$. This function is called a *cross-section* of f with x fixed. The graph of this cross-section is the curve given by intersecting the plane $x = c$ with the graph of $f(x, y)$.

Similarly, the function of one variable given by $f(x, c)$ is a cross-section of f with y fixed. The graph of this function is the curve given by intersecting the plane $y = c$ with the graph of $f(x, y)$.

Each cross-section is a single-variable function, and thus straightforward to graph. By graphing a number of cross sections we can get a good idea what the graph of the whole function looks like.

Example 1.10. Let $f(x, y) = x^2 - y^2$. First we'll take cross-sections holding y constant. We can plot these below in figure 1.4:

Thus we see that the cross-sections holding y constant are parabolas, which start lower and lower the further away we get from the $y = 0$ plane.

We can also take cross-sections holding x constant. We get the similar graphs in figure 1.5:

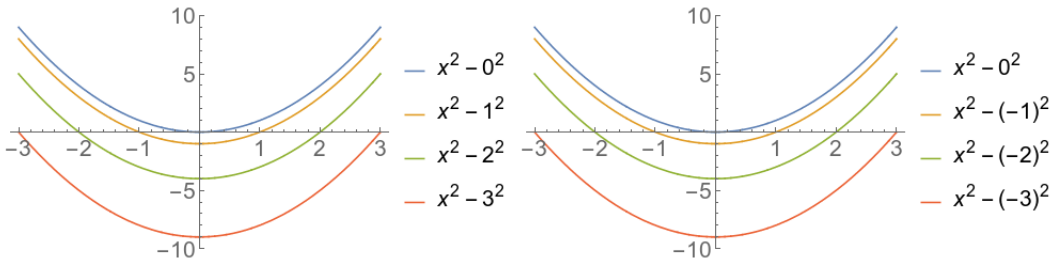


Figure 1.4: Cross sections of $x^2 - y^2$ holding y constant

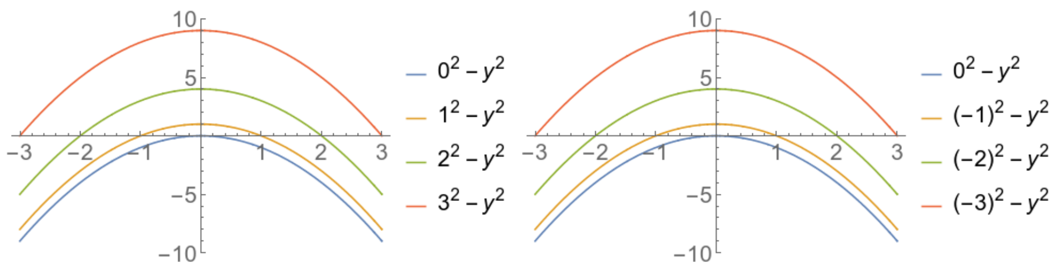


Figure 1.5: Cross sections of $x^2 - y^2$ holding x constant

These show us that holding x constant, we get upside-down parabolas, with the peak being higher and higher the farther we are from the plane $x = 0$.

Putting this together, we can assemble a picture of the real function:

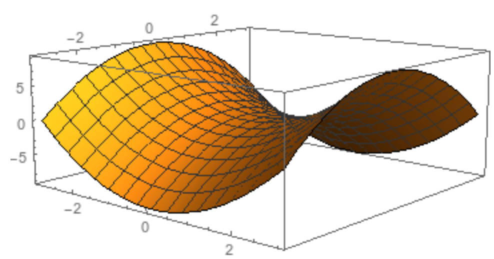


Figure 1.6: The graph of $x^2 - y^2$

Example 1.11. Let $g(x, y) = x^3 + \sin(y)$. We can again take cross sections, holding x and y constant in turn:

From the left, we see that holding x constant, we have a gentle sine wave along the y axis. From the right, we see that holding y constant, x is increasing in a cubic. Putting this information together, we can get a graph for the whole surface:

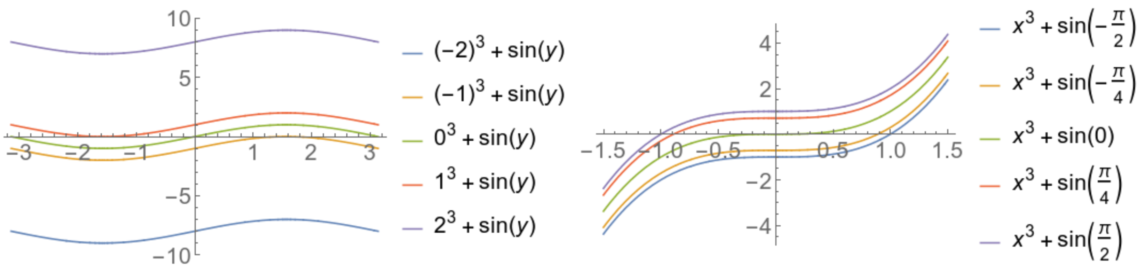


Figure 1.7: Cross sections of $x^3 + \sin(y)$, holding x constant on the left and y constant on the right

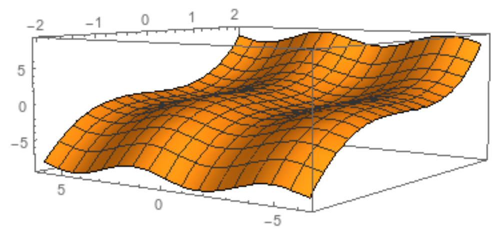


Figure 1.8: The graph of $x^3 + \sin(y)$

1.2.4 Graphing two-variable functions with level sets

Sometimes we want to approach the same question from a different direction (literally!). Instead of holding x constant or y constant, we will hold z constant.

Definition 1.12. If $f(x, y)$ is a function of two variables, then the *level set* of f at level c is the set of all points (x, y) such that $f(x, y) = c$.

A *contour diagram* for f is a graph with several level sets for f at different levels.

Importantly, the level set is not a function, and doesn't need to pass any vertical line tests or anything similar.

Contour diagrams show up commonly in topographical maps.

Example 1.13. The contour plots in figure 1.9 look very similar, but the contour heights make them very different. We can see the corresponding graphs in figure 1.10.

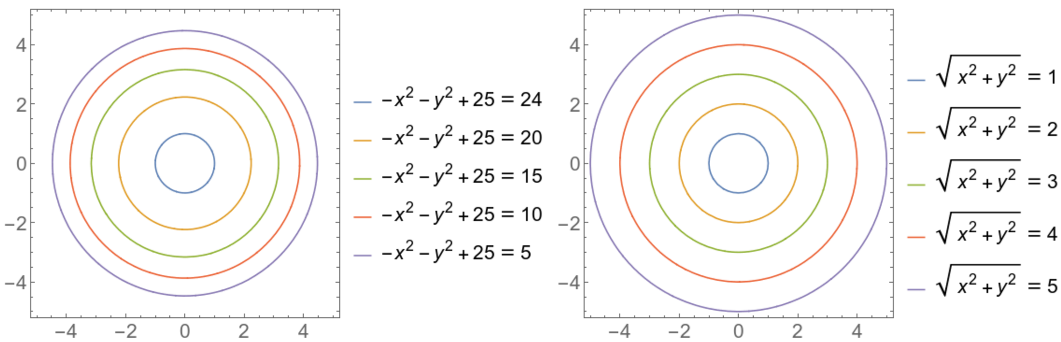


Figure 1.9: Contour diagrams for $f(x, y) = 25 - x^2 - y^2$ and $g(x, y) = \sqrt{x^2 + y^2}$

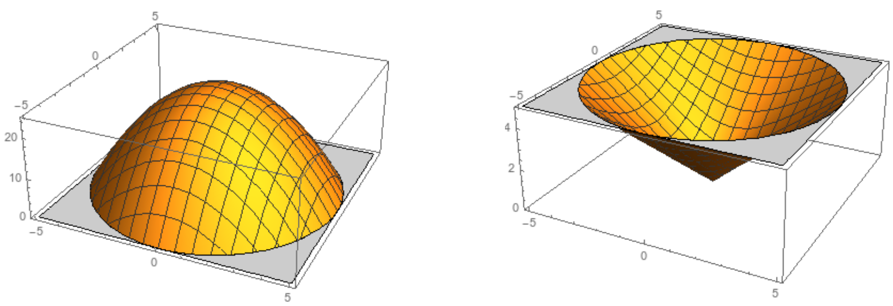


Figure 1.10: The graphs of $f(x, y) = 25 - x^2 - y^2$ and $g(x, y) = \sqrt{x^2 + y^2}$

Example 1.14. We can also draw contour plots for some of our earlier functions. The contour plot for the saddle from example 1.10 and the sine function from example 1.11 appear in figure 1.11.

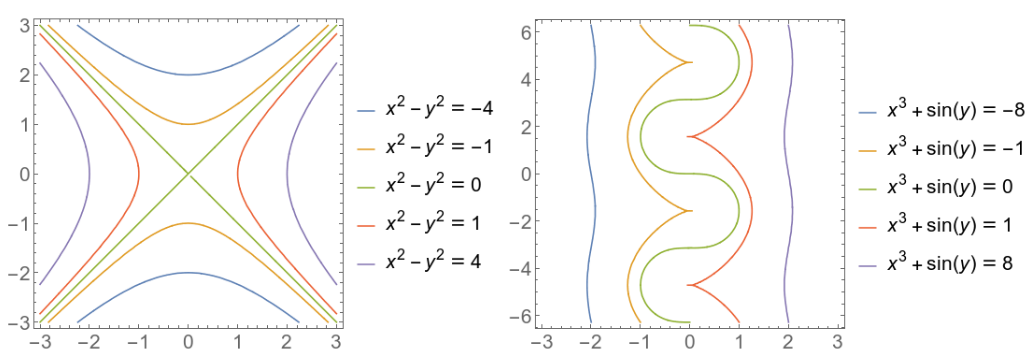


Figure 1.11: Contour plots for $x^2 - y^2$ and $x^3 + \sin(y)$

1.2.5 Graphing three-variable functions with level surfaces

We've now established a few approaches to graphically representing functions of two variables. What can we do with functions of three variables?

Simply graphing the entire function isn't really a plausible solution. As a mathematical object, the graph of a three-variable function as a subset of $\mathbb{R}^4$ is perfectly well defined; but it's almost impossible to draw or visualize these graphs, so they don't help us with our problem of visually representing three-variable functions.

In contrast, cross-sections and level sets are both useful tools. They are much tricker to implement here, because the cross-sections and level sets will themselves be two-variable functions, and thus give us two-dimensional surfaces inside threespace.

Definition 1.15. If $f(x, y, z)$ is a function of two variables, then the *level set* of f at level c is the set of all points (x, y, z) such that $f(x, y, z) = c$.

It's much harder to draw a contour diagram in this case, but we can sort of make an attempt still.

Example 1.16. Find the level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$.

There are no surfaces for $c < 0$, and for $c = 0$ the level surface is a point. For larger c we get a sphere of radius $\sqrt{c}$. Thus the level sets for $c = 1, 4, 9$ are shown in figure 1.12.

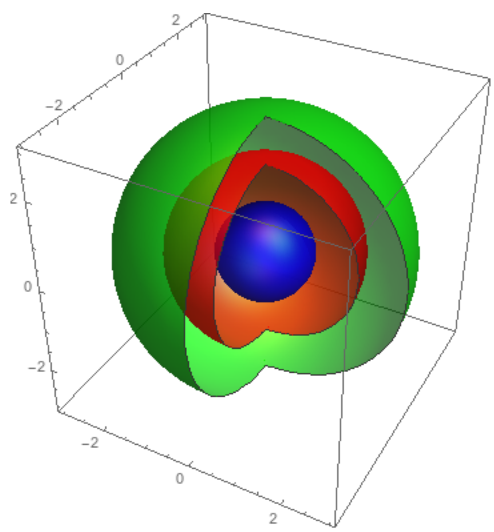


Figure 1.12: Level sets for $x^2 + y^2 + z^2$ at the levels $c = 1, 4, 9$

Example 1.17. We can see the level surfaces of $g(x, y, z) = x^2 + y^2$ and $h(x, y, z) = x + z$, at the levels $1, 2, 3, 4$, in figure 1.13. Thbe level surfaces for g are cylinders of radius $\sqrt{c}$, and the level surfaces of h are all parallel planes.

Example 1.18. We'd like to understand the level surfaces of $f(x, y, z) = x^2 + y^2 - z^2$. These will look different depending on the level of c.

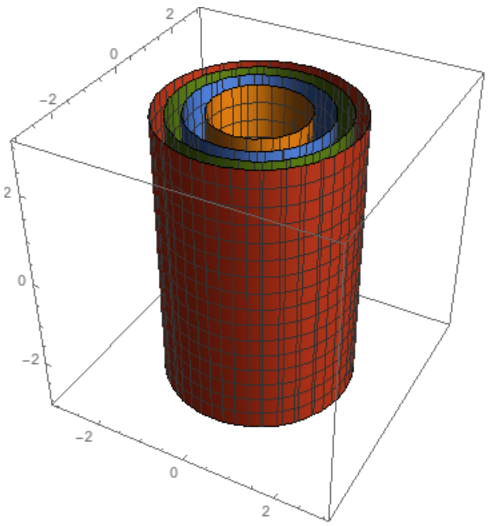


Figure 1.13: The level surfaces of g and h at the levels $1, 2, 3, 4$

It's probably easiest to think about these level surfaces by thinking about their own contour plots as z varies. If $c = 0$, then our equation is $x^2 + y^2 = z^2$. We see that for each z we get a circle of radius z in the plane perpendicular to the z-axis, and in fact at $z = 0$ we have a single point. Stacking these all together gives us two cones.

If c is positive, then we have the equation $x^2 + y^2 = z^2 + c$. Then we see that for each z we get a circle of radius $\sqrt{z^2 + c} > z$, and the radius will always be positive. If instead we take, say, the $x = 0$ cross-section, we get $y^2 - z^2 = c$, which is a hyperbola. The resulting surface is a *hyperboloid of one sheet.*

Finally, if c is negative, we get $x^2+y^2 = z^2+c$, where there is no solution when $z^2+c < 0$. Thus we'll have a stack of increasing radius circles, but it will start at $z = \pm\sqrt{c}$. This surface is a *hyperboloid of two sheets.*

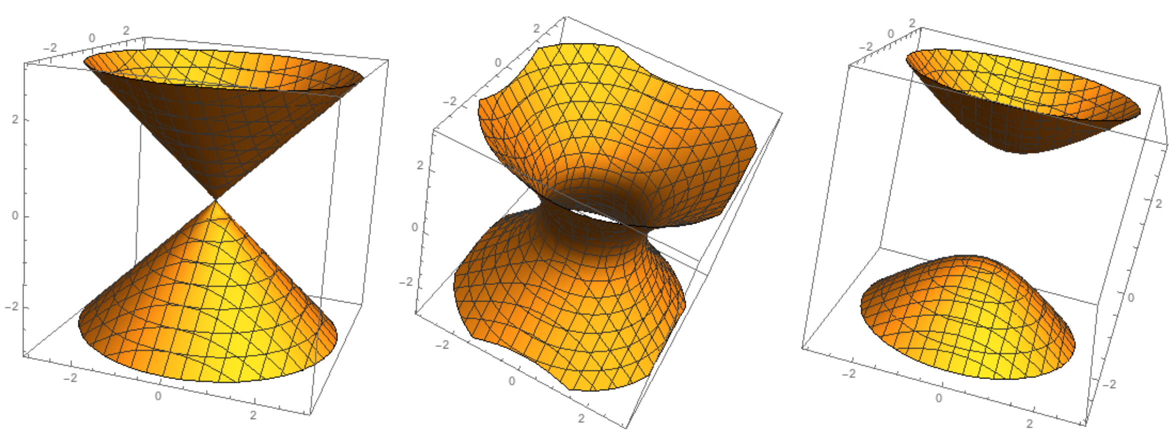


Figure 1.14: Level surfaces of $x^2 + y^2 - z^2$ at the levels $0, 2, -2$

You can see some helpful and common surfaces on page 702 of the textbook, at the end of section 12.5.

Remark 1.19. We've used surfaces to represent the full graph of two-variable functions, and also to represent the level surfaces of three-variable functions. These surfaces are at least somewhat related, and in fact if we have the graph of a function $f(x, y)$, then it is also the level surface at zero of the function $f(x, y) - z$.

Thus every graph of a two-variable function is also a level surface of some three-variable function. The converse, however, is not true; many of the level surfaces we have seen cannot be the graphs of two-variable functions, since they fail the vertical line test.

1.3 Linear Functions

In single-variable calculus, we used the derivative to approximate functions with their tangent lines. We want to do the same sort of approximation for functions of multiple variables; first we need to understand the analogue of a line. We focus on the two-variable case, but our ideas here have obvious generalizations.

Definition 1.20. A *plane* is defined by the equation $z = z_0 + m(x - x_0) + n(y - y_0)$. This plane passes through the point (x_0, y_0, z_0), and we say it has slope m in the x-direction and slope n in the y-direction.

Equivalently, a plane is given by the equation $z = c+mx+ny$ (where $c = z_0 - mx_0 - ny_0$.

The plane is the graph of the *linear function* $f(x, y) = z_0 + m(x - x_0) + n(y - y_0)$.

Remark 1.21. If you have taken linear algebra, you will notice that this is somewhat different from the definition of a linear function given there. A linear function in the linear algebra sense must also pass through the origin, and thus the equation can always be written $f(x, y) = mx + ny$.

Thus technically we have defined an "affine transformation" rather than a linear transformation. But under the same technicality, most lines in single-variable calculus are not linear functions. We'll mostly ignore that here.

Just as a line in the plane is determined by two points, a plane in threespace is determined by three points. The first point gives us (x_0, y_0, z_0), and the other two points tell us the slopes m and n.

Example 1.22. Find the equation of hte plane passing through the points $(1, 0, 1), (1, -1, 3), (3, 0, -1)$.

The first two points have the same x-coordinate, so we can use them to find the y slope. We see that $n = \frac{3-1}{-1-0} = -2$. Then we have

$$\begin{aligned} f(x, y) &= 1 + m(x-1) - 2(y-0) \\ -1 &= 1 + m(3-1) - 2(0-0) = 2m + 1 \end{aligned}$$

so $m = -1$ and the equation for the plane is

$$f(x, y) = 1 - (x-1) - 2y = 2 - x - 2y.$$

We can check our work by plugging all three points back into this equation and confirming that they work.

Example 1.23. Find an equation for the plane going through $(-1, 2, 3)$, $(1, 5, 2)$, and $(3, 4, 1)$.

This time we don't have any conveniently unchanged coordinates. So instead we write

$$\begin{aligned} f(x, y) &= 3 + m(x+1) + n(y-2) \\ 2 &= 3 + m(1+1) + n(5-2) = 3 + 2m + 3n \\ 1 &= 3 + m(3+1) + n(4-2) = 3 + 4m + 2n \\ 2m + 3n &= -1 \\ 4m + 2n &= -2 \\ -4n &= 0 \\ n &= 0 \\ m &= -1/2 \\ f(x, y) &= 3 - \frac{1}{2}(x+1). \end{aligned}$$

The important thing about linear functions is that changes in x and changes in y change the output completely independently. This makes everything about the functions very simple.

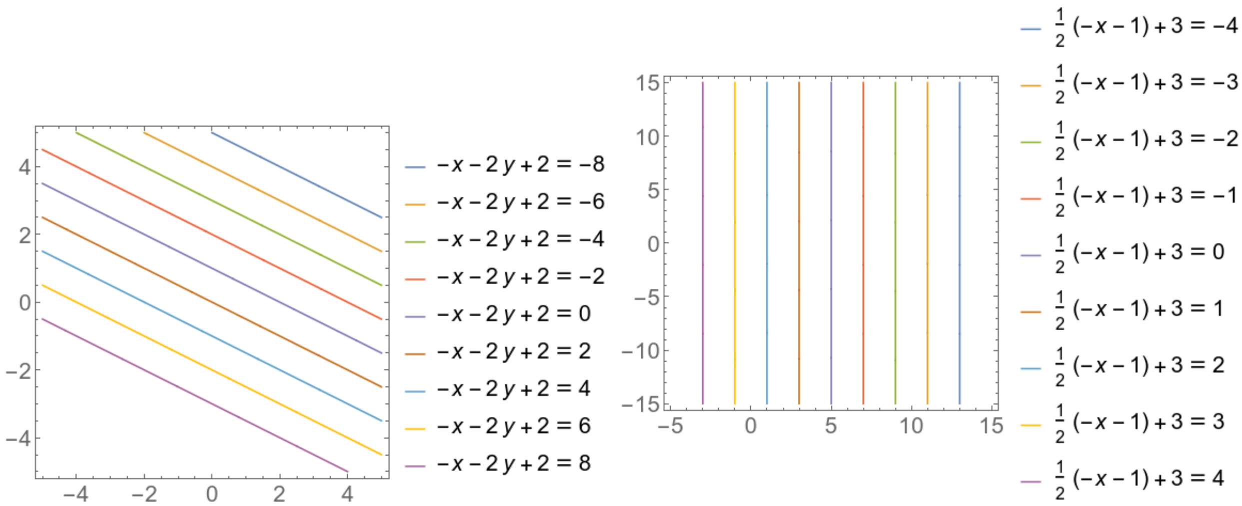


Figure 1.15: Contour diagrams for $f(x, y) = 2 - x - 2y$ and $g(x, y) = 3 - (x + 1)/2$

1.4 Limits and Continuity

In calculus 1, we learned about limits, which tell us in some sense the value a function "should" have at a point—which may or may not be the value it does have, and it may not have a value at all. We can extend the same idea to multivariable functions.

Definition 1.24. If $f : \mathbb{R}^2 \to \mathbb{R}$ is a function, then it has a limit at the point (a, b) of L, and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L,$$

if we can make $f(x, y)$ as close as we want to L, purely by requiring the distance from (x, y) to (a, b) to be small enough (but not zero).

Remark 1.25. Formally we'd write something like: for every $\epsilon > 0$, there is a $\delta > 0$ such that if $\sqrt{(x - a)^2 + (y - b)^2} < \delta$ then $|f(x, y) - L| < \epsilon$. We won't be drilling down into that level of rigor in this class, though. If you want to see more of this kind of thing, take Math 310.

Example 1.26. When the function doesn't do anything weird, limits won't do anything surprising.

$\lim_{(x,y)\to(3,4)} x^2 + y^2 = 3^2 + 4^2 = 25$

$\lim_{(x,y)\to(1,-1)} x^2 - y^2 = 1^2 - (-1)^2 = 0$

Definition 1.27. A function f is *continuous* at a point (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

A function is *continuous on a region* R if it is continuous at each point in R.

If f is not continuous at a point (a, b) then it is *discontinuous* there.

Fact 1.28. *A function defined entirely from algebraic, trigonometric, and exponential functions is continuous anywhere it is defined.*

(No function is ever continuous anywhere it is not defined).

Example 1.29. Let $f(x, y) = \frac{x^2 y}{x^2+y^2}$. This function is continuous everywhere it is defined, which is everywhere except $(0, 0)$. So it's easy to compute, for instance, that $\lim_{(x,y)\to(1,1)} f(x, y) = \frac{1^2 \cdot 1}{1^2+1^2} = \frac{1}{2}$.

Now let's consider $\lim_{(x,y)\to(0,0)} f(x, y)$. We can't just plug $(0, 0)$ in here, so we need to do something else.

First, we can look at the graph and contour diagram of f.

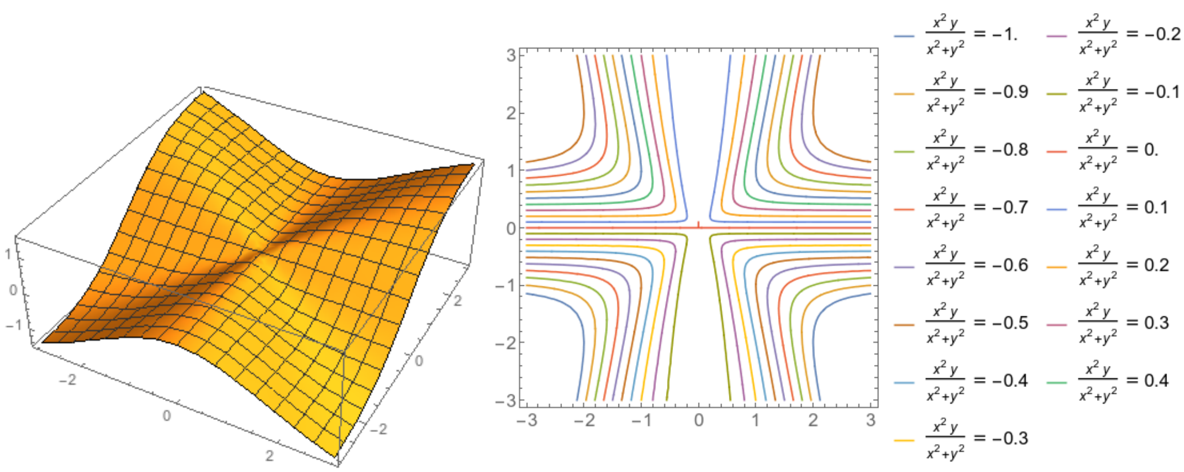


Figure 1.16: The graph and contour plot for $f(x, y) = \frac{x^2 y}{x^2+y^2}$

We see that the function seems to smoothly approach 0 as (x, y) approaches $(0, 0)$, so we suspect the limit is in fact 0.

Informally, we see that the denominator "goes to zero" "twice", while the numerator goes to zero "three times". Thus we would expect the limit to be zero.

If we want to be more rigorous, we calculate the distance between $f(x, y)$ and the guessed limit 0. Then we have

$$|f(x, y) - L| = \left| \frac{x^2 y}{x^2 + y^2} - 0 \right| = \left| \frac{x^2}{x^2 + y^2} \right| |y| \leq |y| \leq \sqrt{x^2 + y^2}.$$

Thus the distance between $f(x, y)$ and 0 is less than the distance between (x, y) and $(0, 0)$. Clearly by making (x, y) closer to $(0, 0)$ we can make $f(x, y)$ as close as we want to 0.

Since the limit exists, we can extend this function to be continuous at the origin: the function

$$f_f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$.

Example 1.30. Now let's consider the similar function $g(x,y) = \frac{x^2}{x^2+y^2}$. Like in example 1.29, this is continuous everywhere it is defined, which is everywhere except at $(0,0)$.

But at $(0,0)$ things are tricker. The graph has a noticeable spike, and the contour plot looks terrible near $(0,0)$, with all the contours converging onto that single point.

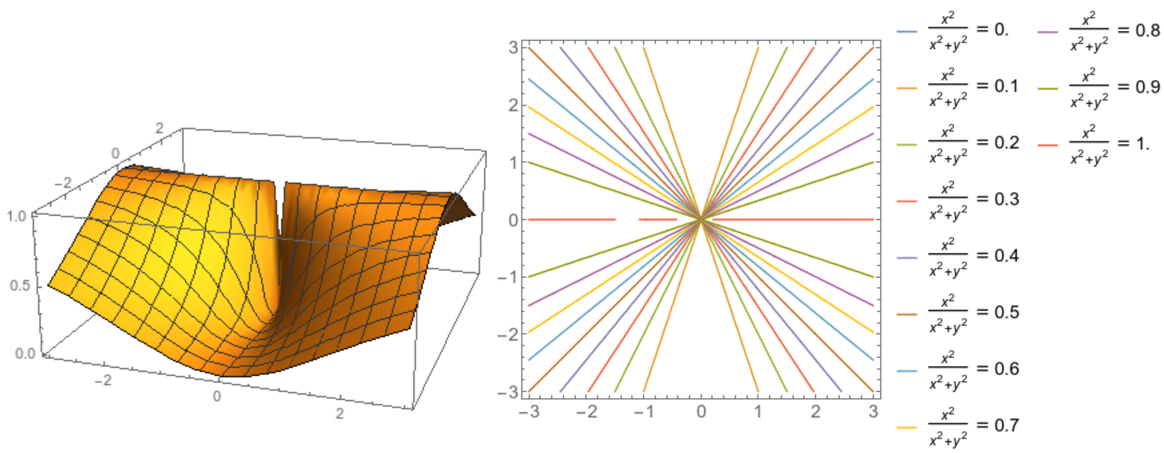


Figure 1.17: The graph and contour plot for $g(x,y) = \frac{x^2}{x^2+y^2}$

The informal algebraic argument we gave before doesn't help: both the top and the bottom go to zero "twice". So this doesn't help us find any limit.

Formally, we want to show that no limit exists, so we want to show that you can be as close to $(0,0)$ as you want and still get very different answers for $g(x,y)$.

So first let's consider points that look like $(a,0)$. Then $g(a,0) = \frac{a^2}{a^2+0^2} = 1$. Since a can be as small as we want, this tells us that we can be as close to the origin as we want and have $g(x,y) = 1$.

But this doesn't mean the limit is 1! As an example, take points that look like $(0,b)$. Then $g(0,b) = \frac{0^2}{0^2+b^2} = 0$. Since b can be anything, this also tells us that we can be as close to the origin as we want, and have $g(x,y) = 0$. Thus no limit exists.

In fact, by approaching from the right direction, we can get any value between 0 and 1. And we can see this behavior both in the graph (which has an abrupt spike or dip near the origin), and in the contour plot (which shows us different directions of approach, and the values they will give).

We just saw that we can show that limits don't exist by approaching the same point from different directions. This should remind you of the one-variable case, where we might check the right- and left-sided limits and show they differ.

But the multivariable case is considerably more complex, because there are infinitely many directions. (In fact it's more complicated than that: there are functions that have a consistent limit as long as you approach along any straight-line path, but that break down when you approach along the right curve).

But if we want to understand this better, we'll need a language for talking about directions.

2 Vectors and Geometry

2.1 Vectors in Space

Definition 2.1 (informal)**.** A *vector* is a mathematical object that encodes both direction and size or *magnitude*. We notate vectors with an arrow over them, as in $\vec{v}$.

A *displacement vector* from one point to another is an arrow with its tail at one point and its head at the other. It gives the distance between the two points, and the direction from the first point to the second point. The vector from the point P to the point Q is written $\overrightarrow{PQ}$.

Notice that it is possible for $\overrightarrow{PQ}$ to be the same vector as $\overrightarrow{AB}$ even if A and B are different points from P and Q. The vector encodes the distance and direction, but not the specific points.

A displacement vector whose tail is at the origin is called a *position vector*.

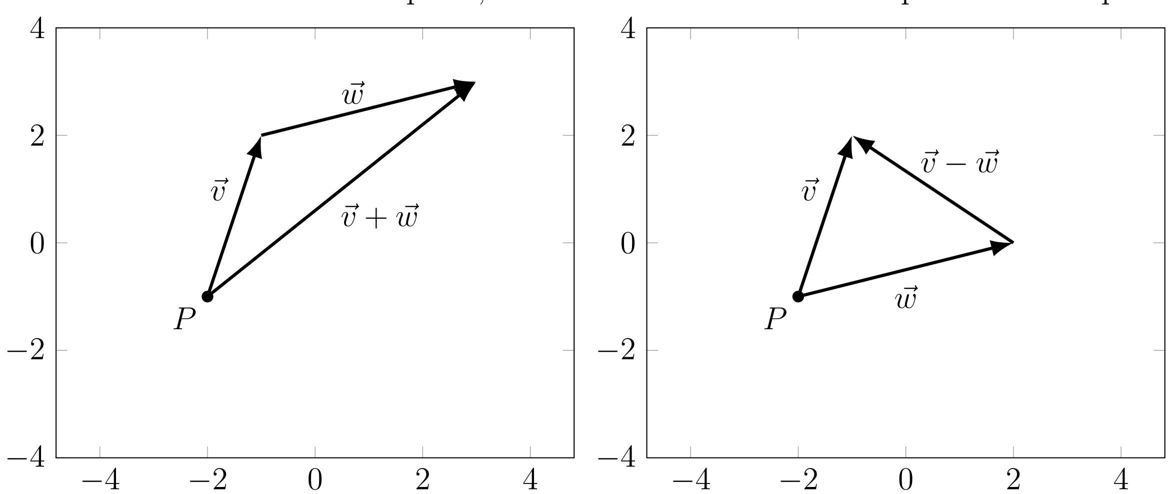
A quantity that has size but no direction is called a *scalar*.

Remark 2.2. We have a distinction between vectors and scalars in single-variable calculus, but we can mostly avoid thinking too hard about it since there are only two possible directions. Vectors show up, for instance, in the idea of one-sided limits.

We can do arithmetic on vectors. Adding vectors, geometrically, represents the displacement of following one vector and then the other, putting them tail-to-tip.

Definition 2.3. The *sum* $\vec{v} + \vec{w}$ of two vectors is the vector given by following $\vec{v}$ and then following $\vec{w}$. Thus if $\vec{v} = \overrightarrow{PQ}$ and $\vec{w} = \overrightarrow{QR}$ then $\vec{v} + \vec{w} = \overrightarrow{PR}$.

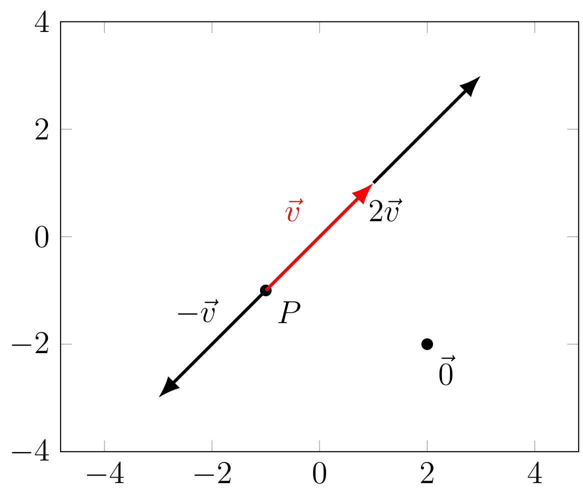
The *difference* of two vectors $\vec{w} - \vec{v}$ is the vector that, when added to $\vec{v}$, gives $\vec{w}$. That is, if $\vec{v}$ and $\vec{w}$ have the same base point, $\vec{w} - \vec{v}$ is the arrow from the tip of $\vec{v}$ to the tip of $\vec{w}$.



Scalar multiplication represents stretching a vector, and also possibly reversing its direction.

Definition 2.4. If λ is a scalar (real number), and $\vec{v}$ is a (displacement) vector, then the *scalar multiple* of $\vec{v}$ by λ is a vector stretched by a factor of $|\lambda|$. It points in the same direction as $\vec{v}$ if $\lambda > 0$ and in the opposite direction if $\lambda < 0$.

If $\lambda = 0$ then $\lambda\vec{v}$ is the *zero vector* $\vec{0}$, which has zero magnitude. This vector is the same regardless of direction, and corresponds to $\overrightarrow{PP}$ for any point P.



Vector arithmetic has a bunch of useful properties.

Fact 2.5. *Let $\vec{u}, \vec{v}, \vec{w}$ be vectors, and r, s be scalars. Then:*

1. *(Additive commutativity)* $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. *(Additive associativity)* $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3. *(Additive identity)* $\vec{u} + \vec{0} = \vec{u}$.
4. *(Additive inverses)* $\vec{u} + (-1)\vec{v} = \vec{u} - \vec{v}$. *We write* $(-1)\vec{v} = -\vec{v}$.
5. *(Distributivity)* $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$
6. *(Distributivity)* $(r + s)\vec{u} = r\vec{u} + s\vec{u}$
7. *(Multiplicative associativity)* $r(s\vec{u}) = (rs)\vec{u}$
8. *(Multiplicative Identity)* $1\vec{u} = \vec{u}$
9. *(Zero Length)* $0\vec{v} = \vec{0}$.

Remark 2.6. In linear algebra we say this list of properties defines a *vector space*.

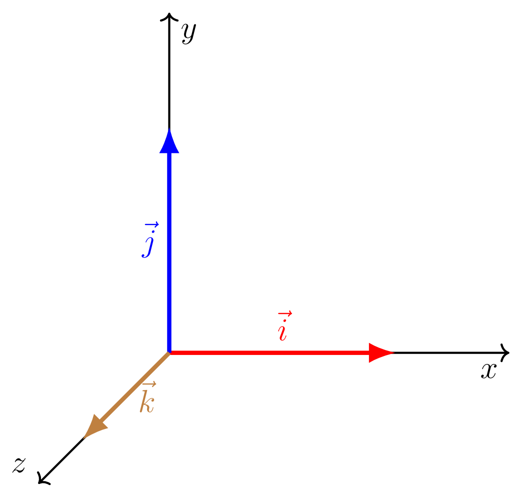
This seems like a long list of properties, but most of them are things you were probably assuming were true anyway. We could prove any of these properties by drawing vectors or working through some algebra, but we'll pass over that because it's boring and not very enlightening. .

2.2 Vector Components and Algebra

So far we've defined vectors and stated a bunch of properties they have, but all of this has been stated entirely in terms of geometry. We'd now like to establish the other side of our duality and express vectors in algebraic terms.

We first define three "standard" vectors.

Definition 2.7. We define the vector $\vec{i} = \overrightarrow{(0,0,0)(1,0,0)}$ to be the vector of length 1 in the positive x direction. Similarly we define $\vec{j}$ to be the vector of length 1 in the positive y direction, and $\vec{k}$ to be the vector of length 1 in the positive z direction.



Remark 2.8. In linear algebra terms, these three vectors are a *basis* for $\mathbb{R}^3$. In fact, these three vectors are the standard basis vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$.

We can take any vector in $\mathbb{R}^3$ and express it in terms of these vectors. If we know how far to go in the x direction, how far in the y direction, and how far in the z direction, then we know exactly where to go, so adding and multiplying these vectors can give us everything we need to identify any possible vector.

Definition 2.9. If $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ where the v_i are scalars, then we say that we have *resolved* $\vec{v}$ into components, and the summands are the *components* of $\vec{v}$.

We will sometimes write $\vec{v} = (v_1, v_2, v_3)$ or $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ or $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. But in this course we will usually find it more convenient to use the $\vec{i}, \vec{j}, \vec{k}$ notation.

Example 2.10. Let $P = (1, 3, 2)$ and $Q = (2, -4, 1)$. Then the vector $\overrightarrow{PQ} = \vec{i} - 7\vec{j} - \vec{k}$.

The vector $2\vec{i} - 14\vec{j} - 2\vec{k}$ is parallel to $\overrightarrow{PQ}$ since it is a scalar multiple of $\overrightarrow{PQ}$. But the vector $\vec{i} + 7\vec{j} + \vec{k}$ is not.

This sort of resolution or decomposition makes it easy to do vector arithmetic algebraically.

Example 2.11. If $\vec{v} = \overrightarrow{PQ} = \vec{i} - 7\vec{j} - \vec{k}$ and $\vec{u} = 2\vec{i} - 3\vec{j} + 2\vec{k}$, then $\vec{u} + \vec{v} = 3\vec{i} - 10\vec{j} + \vec{k}$.

We have $3\vec{v} = 3\vec{i} - 21\vec{j} - 3\vec{k}$.

2.3 Angles, Magnitudes and the Dot Product

We talk about vectors as having a direction and a magnitude. We've talked about this geometrically in terms of arrows; we've algebraically resolved them into components. But we can also numerically specify a direction in terms of angles, and then simply give a magnitude.

2.3.1 Vector Magnitudes

Definition 2.12. The *magnitude* of a vector $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ is $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

If $\|\vec{v}\| = 1$ then we say that $\vec{v}$ is a *unit vector*.

Example 2.13. Let $\vec{v} = 3\vec{i} + 1\vec{j} - 2\vec{k}$ and $\vec{w} = 2\vec{i} - \vec{j} + 4\vec{k}$. Then $\|\vec{v}\| = \sqrt{3^2 + 1^1 + 2^2} = \sqrt{14}$ and $\|\vec{w}\| = \sqrt{2^2 + 1^2 + 4^2} = \sqrt{21}$.

If we want unit vectors in the same direction, we can take the vectors

$$\frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{v}}{\sqrt{14}} = \frac{3}{\sqrt{14}}\vec{i} + \frac{1}{\sqrt{14}}\vec{j} - 2\frac{2}{\sqrt{14}}\vec{k}$$
$$\frac{\vec{w}}{\|\vec{w}\|} = \frac{\vec{w}}{\sqrt{21}} = \frac{2}{\sqrt{21}}\vec{i} - \frac{1}{\sqrt{21}}\vec{j} + \frac{4}{\sqrt{21}}\vec{k}.$$

Example 2.14. Let's find a unit vector based at the point (x, y, z) that points directly away from the origin.

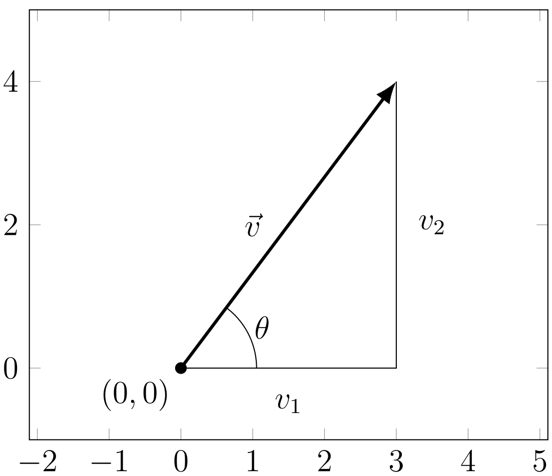
We need a vector that points in the same direction as $x\vec{i} + y\vec{j} + z\vec{k}$, but has unit length. So we compute

$$\|x\vec{i} + y\vec{j} + z\vec{k}\| = \sqrt{x^2 + y^2 + z^2}$$

so our unit vector is

$$\frac{x}{\sqrt{x^2+y^2+x^2}}\vec{i}+\frac{y}{\sqrt{x^2+y^2+x^2}}\vec{j}+\frac{z}{\sqrt{x^2+y^2+x^2}}\vec{k}.$$

From the Pythagorean theorem we can see that the magnitude is actually the length of the arrow corresponding to the vector. In two dimensions, we see that the length of the vector $v_1\vec{i}+v_2\vec{j}$ is $\sqrt{v_1^2+v_2^2}$; the same argument works in three (or more) dimensions by repeatedly applying the Pythagorean theorem.



We can also use trigonometry to relate the components v_1 and v_2 with the magnitude $\|\vec{v}\|$ and the angle θ.

Proposition 2.15. *If a vector $\vec{v}=v_1\vec{i}+v_2\vec{j}$ makes an angle of θ with the x axis, then:*

- $v_1=\|\vec{v}\|\cos\theta$
- $v_2=\|\vec{v}\|\sin\theta$
- $\theta=\arctan {}^{v_2}/_{v_1}$.

Example 2.16. Let $\vec{v}=3\vec{i}+4\vec{j}$. Then $\|\vec{v}\|=\sqrt{3^2+4^2}=5$, and $\vec{v}$ makes an angle of approximately .92 with the x axist.

Let $\vec{w}$ have length 7 and make an angle of $\pi/6$ with the x axis. Then

$$\vec{w}=7\cos(\pi/6)\vec{i}+7\sin(\pi/6)\vec{j}=\frac{7\sqrt{3}}{2}\vec{i}+\frac{7}{2}\vec{j}\approx 6.06\vec{i}+3.5\vec{j}.$$

2.3.2 The Dot Product

To think more about angles and magnitudes, we want to consider the dot product.

Definition 2.17. If $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$ then the *dot product* $\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + v_3w_3$.

Example 2.18. Let $\vec{v} = 3\vec{i} + 1\vec{j} - 2\vec{k}$ and $\vec{w} = 2\vec{i} - \vec{j} + 4\vec{k}$. Then $\vec{v} \cdot \vec{w} = 6 - 1 - 8 = -3$.

Notice the dot product takes in two vectors and gives a scalar. Also notice that $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$.

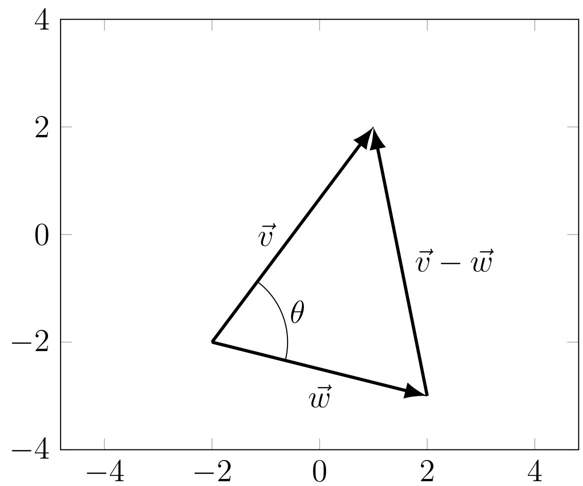
Proposition 2.19. *If $\vec{u}, \vec{v}, \vec{w}$ are vectors and λ is a scalar, then:*

- $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$
- $\vec{v} \cdot (\lambda\vec{w}) = \lambda(\vec{v} \cdot \vec{w}) = (\lambda\vec{v}) \cdot \vec{w}$
- $(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u}$.

We can use the dot product to find the angles between two vectors.

Proposition 2.20. *If $\vec{v}, \vec{w}$ are vectors, and θ is the angle between them, then $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta$.*

Proof. We can prove this using the law of cosines, which states that if a triangle has sides of length a, b, c, and the angle opposite side c has measure θ, then $c^2 = a^2 + b^2 - 2ab\cos\theta$. (Notice that if c is the hypotenuse of a right triangle, then $\cos\theta = \cos\pi/2 = 0$ and we recover the pythagorean theorem).



So form a triangle with sides $\vec{v}, \vec{w}$, and $\vec{v}-\vec{w}$. Then the law of cosines gives us

$$\begin{aligned}
\|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos\theta &= \|\vec{v}-\vec{w}\|^2 \\
\left(v_1^2+v_2^2+v_3^2\right) + \left(w_1^2+w_2^2+w_3^2\right) - 2\|\vec{v}\|\|\vec{w}\|\cos\theta &= (v_1-w_1)^2 + (v_2-w_2)^2 + (v_3-w_3)^2 \\
&= v_1^2 - 2v_1w_1 + w_1^2 \\
&\quad + v_2^2 - 2v_2w_2 + w_2^2 \\
&\quad + v_3^2 - 2v_3w_3 + w_3^2
\end{aligned}$$

and subtracting the squared terms from both sides gives

$$\begin{aligned}
-2\|\vec{v}\|\|\vec{w}\|\cos\theta &= -2v_1w_1 - 2v_2w_2 - 2v_3w_3 \\
\|\vec{v}\|\|\vec{w}\|\cos\theta &= v_1w_1 + v_2w_2 + v_3w_3 = \vec{v}\cdot\vec{w}.
\end{aligned}$$

□

Example 2.21. Suppose $\|\vec{v}\| = 3$ and $\|\vec{w}\| = 5$. What is the maximum possible dot product, and when does this happen? What is the minimum possible dot product, and when does this happen? When is the dot product 0?

The dot product is $\vec{v}\cdot\vec{w} = 15\cos\theta$. So the dot product is maximized when $\cos\theta$ is maximized, which happens when $\theta = 0$; thus when the vectors point in the same direction, their dot product is 15. Similarly, when $\theta = \pi$ the vectors point in opposite directions, and their dot product is minimized at -15.

The dot product is 0 when $\cos\theta = 0$, which happens when $\theta = \pi/2$. Thus this occurs when the vectors are at a right angle.

Definition 2.22. We say that two vectors $\vec{v}$ and $\vec{w}$ are *perpendicular* or *orthogonal* if $\vec{v}\cdot\vec{w} = 0$. This corresponds to the two vectors forming a right angle.

If we know the angle between two vectors, we can use this to compute the dot product; but more often we use the dot product to compute the angle between two vectors.

Example 2.23. Let $\vec{v} = 3\vec{i} + 2\vec{j}$ and $\vec{w} = 2\vec{i} - \vec{j}$. Then $\vec{v}\cdot\vec{w} = 6 - 2 = 4$. We see that $\|\vec{v}\| = \sqrt{13}$ and $\|\vec{w}\| = \sqrt{5}$ so

$$\theta = \arccos\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right) = \arccos\left(\frac{4}{\sqrt{65}}\right) \approx 1.05.$$

Let $\vec{v} = 3\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{w} = 2\vec{i} - \vec{j} + 4\vec{k}$. Then $\vec{v}\cdot\vec{w} = 6 - 2 - 4 = 0$. Thus the angle between $\vec{v}$ and $\vec{w}$ is

$$\arccos\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right) = \arccos(0) = \pi/2.$$

We see that $\vec{v}$ and $\vec{w}$ are orthogonal.

2.3.3 The Dot Product, Lines, and Planes

We can use the dot product and angles to understand lines and planes better.

First we consider a plane. We can think of a given plane as being the set of all lines thought a given point perpendicular to a given line. We want to rephrase this idea in terms of vectors.

Definition 2.24. If $\vec{n}$ is perpendicular to a plane—that is, perpendicular to any vector between two points in the plane—then we say that $\vec{n}$ is a *normal vector* to the plane.

If $P = (x_0, y_0, z_0)$ is a point on the plane, then the plane consists of all points $Q = (x, y, z)$ such that $\overrightarrow{PQ}$ is perpendicular to $\vec{n} = a\vec{i}+b\vec{j}+c\vec{k}$. Since $\overrightarrow{PQ} = (x-x_0)\vec{i}+(y-y_0)\vec{j}+(z-z_0)\vec{k}$, we see that the plane is the set of points satisfying

$$\begin{aligned} \vec{n} \cdot \overrightarrow{PQ} &= 0 \\ a(x-x_0) + b(y-y_0) + c(z-z_0) &= 0. \end{aligned}$$

Notice that this is equivalent to the equation for a plane we gave in section 1.3. In particular, if we take $d = ax_0 + by_0 + cz_0$ then our equation is

$$ax + by + cz = d.$$

Example 2.25. Find an equation for the plane perpendicular to $\vec{n} = \vec{i} - 2\vec{j} + \vec{k}$ and passing through the point $(1, 5, 2)$.

The equation is

$$(x-1) - 2(x-5) + (x-2) = 0.$$

Example 2.26. Find a normal vector to the plane given by the equation

$$5x - 2y + 3z = 32.$$

$\vec{n} = 5\vec{i} - 2\vec{j} + 3\vec{k}$.

Find a normal vector to the plane given by

$$z = 3(x-1) + 2(y-5) + 2.$$

We rewrite the equation to be $0 = 3(x-1) + 2(y-5) - z + 2$. Then we see that $\vec{n} = 3\vec{i} + 2\vec{j} - \vec{k}$.

We can also use the dot product to project a vector onto a line. In fact this is probably the best way to understand what the dot product "really means".

Earlier we used trigonometry to resolve a vector into its $\vec{i}$ and $\vec{j}$ components. But we can do the same thing to a coordinate axis defined by any vector.

Definition 2.27. Let $\vec{v}, \vec{u}$ be two non-zero vectors. We define the *projection* of $\vec{v}$ onto $\vec{u}$ to be

$$\vec{v}_{\text{parallel}} = \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2}\vec{u} = \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u}.$$

In particular, if $\vec{u}$ is a unit vector, then $\vec{v}_{\text{parallel}} = (\vec{v} \cdot \vec{u})\vec{u}$.

We define the *orthogonal complement* of $\vec{v}$ on $\vec{u}$ to be $\vec{v}_{\text{perp}} = \vec{v}_\perp = \vec{v} - \vec{v}_{\text{parallel}}$.

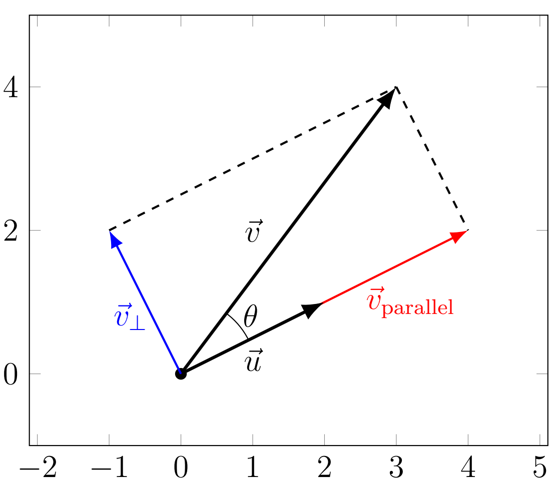
Proposition 2.28. *Let $\vec{v}, \vec{u}$ be non-zero vectors. Then:*

- *$\vec{v}_{parallel}$ is parallel to $\vec{u}$.*
- *$\vec{v}_\perp \cdot \vec{u} = 0$.*

Thus $\vec{v}_{parallel}$ represents the part of $\vec{v}$ that goes in the same direction as $\vec{u}$, and $\vec{v}_\perp$ represents the remainder.

Proof. We'll prove the first fact assuming the angle θ between $\vec{v}$ and $\vec{u}$ satisfies $0 \leq \theta \leq \pi/2$. The proof for $\pi/2 \leq \theta \leq \pi$ is basically the same but involves a slightly different picture. We never need to consider $\theta > \pi$ since we can just look at the angle going around the other way.

Consider the following diagram:



We see by trigonometry that $\|\vec{v}_{\text{parallel}}\| = \|\vec{v}\| \cos\theta$. Since we know that $\vec{v}_{\text{parallel}}$ has the same direction as $\vec{u}$, we see that

$$\begin{aligned}
\vec{v}_{\text{parallel}} &= \|\vec{v}\| \cos\theta \frac{\vec{u}}{\|\vec{u}\|} \\
&= \|\vec{v}\|\|\vec{u}\| \cos\theta \frac{1}{\|\vec{u}\|^2}\vec{u} \\
&= \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}
\end{aligned}$$

by proposition 2.28.

We can now prove the second fact with simple algebra. We have

$$\begin{aligned}
\vec{v}_{\perp} \cdot \vec{u} = (\vec{v} - \vec{v}_{\text{parallel}}) \cdot \vec{u} &= \vec{v}\cdot\vec{u} - \vec{v}_{\text{parallel}} \cdot \vec{u} \\
&= \vec{v}\cdot\vec{u} - \left(\frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}\right)\cdot\vec{u} \\
&= \vec{v}\cdot\vec{u} - \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}(\vec{u}\cdot\vec{u}) = \vec{v}\cdot\vec{u} - \vec{v}\cdot\vec{u} = 0.
\end{aligned}$$

$\square$

Example 2.29. Let's decompose the vector $\vec{v} = 3\vec{i} + 4\vec{j}$ with respect to $\vec{u} = 2\vec{i} + \vec{j}$.

We compute

$$\begin{aligned}
\vec{v}_{\text{parallel}} &= \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{6+4}{4+1}(2\vec{i}+\vec{j}) = 4\vec{i} + 2\vec{j} \\
\vec{v}_{\perp} &= \vec{v} - \vec{v}_{\text{parallel}} = 3\vec{i} + 4\vec{j} - 4\vec{i} - 2\vec{j} = -\vec{i} + 2\vec{j}.
\end{aligned}$$

We can see that $\vec{v}_{\text{parallel}}$ is indeed parallel to $\vec{u}$, and that $\vec{v}_{\perp} + \vec{v}_{\text{parallel}} = \vec{v}$. We also check that

$$\vec{v}_{\perp} \cdot \vec{u} = (-\vec{i} + 2\vec{j}) \cdot (2\vec{i} + \vec{j}) = -2 + 2 = 0.$$

Thus $\vec{v}_{\perp} \perp \vec{u}$ as we wanted.

Example 2.30. Let's decompose $\vec{v} = 2\vec{i} - 3\vec{j} + \vec{k}$ onto $\vec{u} = \vec{i} + \vec{j} - 2\vec{k}$.

$$\begin{aligned}
\vec{v}_{\text{parallel}} &= \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = 2 - 3 - 21 + 1 + 4(\vec{i} + \vec{j} - 2\vec{k}) = -\vec{i}/2 - \vec{j}/2 + \vec{k} \\
\vec{v}_{\perp} &= \vec{v} - \vec{v}_{\text{parallel}} = 2\vec{i} - 3\vec{j} + \vec{k} + \vec{i}/2 + \vec{j}/2 - \vec{k} = 5\vec{i}/2 - 5\vec{j}/2.
\end{aligned}$$

And we check that

$$\vec{u} \cdot \vec{v}_{\perp} = (\vec{i} + \vec{j} - 2\vec{k}) \cdot (5\vec{i}/2 - 5\vec{j}/2) = 5/2 - 5/2 = 0.$$

One common application of this idea is *work* in physics. The work done by a force on an object is the force applied in the direction of motion, times the distance traveled. But if the force is not in the direction of motion (which happens, for instance, if the object is already moving when force is applied), then we can use the dot product to calculate the work.

Example 2.31. Suppose a force of 10 Newtons is applied in the $\vec{i} + \vec{j}$ direction to an object that moves along displacement $\vec{d} = 3\vec{i} - 2\vec{j}$. What is the work done?

We can decompose the force into the force into the force $\vec{F}_{\text{parallel}}$ in the direction of the displacement, and the force $F_\perp$ perpendicular to the direction.

But in fact we know that $\|\vec{F}_{\text{parallel}}\| = \|\vec{F}\| \cos\theta$, so

$$W = \|\vec{F}_{\text{parallel}}\|\|\vec{d}\| = \|\vec{F}\| \cos\theta \|\vec{d}\| = \vec{F} \cdot \vec{d}.$$

Thus we can use the dot product to compute the work. So we have

$$W = \vec{F} \cdot \vec{d} = 15\sqrt{2} - 10\sqrt{2} = 5\sqrt{2}.$$

This concept will come up a lot when we start studying integrals.

2.4 The Cross Product

The dot product takes in two vectors and gives us a scalar. But often we want to take two vectors and get another vector. The cross product answers a very specific question: given two vectors, find a vector that is perpendicular to both of them.

Since there are two directions "perpendicular to both vectors", we choose one based on the *right-hand rule*, which says that if you point the fingers of your right hand towards $\vec{u}$ and curl your fingers towards $\vec{v}$ then your thumb will point in the direction of $\vec{u} \times \vec{v}$.

Definition 2.32. Let $\vec{u}$ and $\vec{v}$ be vectors, and let θ be the angle between them. Then the *cross product* of $\vec{u}$ and $\vec{v}$ is given by

$$\vec{u} \times \vec{v} = (\|\vec{u}\|\|\vec{v}\| \sin\theta)\vec{n}$$

where $\vec{n}$ is a unit normal vector to the plane containing $\vec{u}$ and $\vec{v}$, pointing in the direction given by the right-hand rule.

Remark 2.33. If $\vec{u}$ and $\vec{v}$ point in the same direction, then there is more than one plane through both of them and the vector $\vec{n}$ isn't clearly defined. But in this case $\sin\theta = \sin(0) = 0$ so $\vec{u} \times \vec{v} = \vec{0}$, and we don't need to worry about the direction.

Fact 2.34. *let $\vec{u}$ and $\vec{v}$ be vectors. Then*

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\vec{i} + (u_3v_1 - u_1v_3)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}.$$

We often remember this definition by writing it as a determinant:

$$\vec{u} \times \vec{v} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Example 2.35. Let $\vec{u} = \vec{i} - 3\vec{j} + \vec{k}$ and $\vec{v} = -2\vec{i} + \vec{j} + 2\vec{k}$. Then

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & 1 \\ -2 & 1 & 2 \end{vmatrix} = (-6-1)\vec{i} + (-2-2)\vec{j} + (1-6)\vec{k} = -7\vec{i} - 4\vec{j} - 5\vec{k}.$$

We can check that this is perpendicular to both $\vec{u}$ and $\vec{v}$.

The cross product makes it really easy to find equations for planes in threespace. If you have two vectors parallel to the plane, the cross product will give you a normal vector, which is all you need to write down the equation for the plane.

Example 2.36. Find an equation for the plane parallel to the vectors $\vec{u} = \vec{i} + \vec{j}$ and $\vec{v} = 5\vec{i} - \vec{j} + 3\vec{k}$ through the point $(5, 1, 3)$.

We compute

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 5 & -1 & 3 \end{vmatrix} = (3-0)\vec{i} + (0-3)\vec{j} + (-1-5)\vec{k} = 3\vec{i} - 3\vec{j} - 6\vec{k}.$$

We can double-check that this is perpendicular to both our original vectors.

For the equation of the plane we need a point, which we have, and a normal vector, which is $\vec{u} \times \vec{v}$. Thus the equation for the plane is

$$0 = 3(x-5) - 3(y-1) - 6(z-3).$$

Example 2.37. Use the cross product to find an equation for the plane containing the three points $(1, 4, 2), (5, 1, 1), (-2, 1, 7)$.

We see that the vectors from the first point to the other two are $4\vec{i} - 3\vec{j} - \vec{k}$ and $-3\vec{i} - 3\vec{j} + 5\vec{k}$. Thus we compute

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -3 & -1 \\ -3 & -3 & 5 \end{vmatrix} = (-15-3)\vec{i} + (3-20)\vec{j} + (-12-9)\vec{k} = -18\vec{i} - 17\vec{j} - 21\vec{k}.$$

Thus an equation for the plane is

$$0 = 18(x-1) + 17(y-4) + 21(z-2).$$

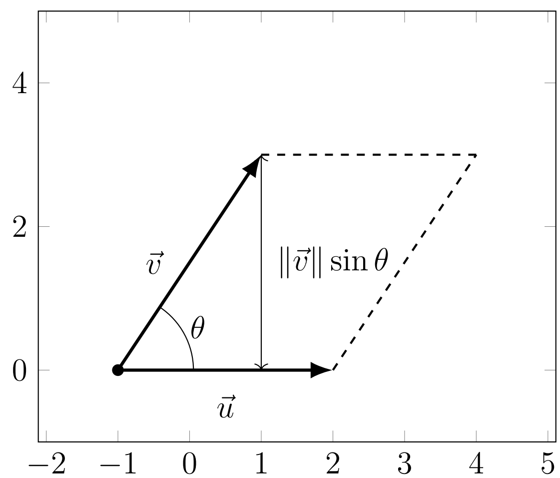
Proposition 2.38. *If $\vec{u}, \vec{v}, \vec{w}$ are vectors and λ is a scalar, then:*

- $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$.
- $(\lambda\vec{u}) \times \vec{v} = \lambda(\vec{u} \times \vec{v}) = \vec{y} \times (\lambda\vec{v})$.
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.

We can also interpret the cross product as measuring an area.

Proposition 2.39. *Let $\vec{u}, \vec{v}$ be two vectors in a plane, and let θ be the angle between them. Then the area of the parallelogram with $\vec{u}$ and $\vec{v}$ as two sides is $\|\vec{u} \times \vec{v}\|$.*

Proof. The area of a parallelogram is the length of the base times the height.



The length of the base is $\|\vec{u}\|$, and the height is $\|\vec{v}\| \sin\theta$ by basic trigonometry. Thus the area of the parallelogram is $\|\vec{v}\|\|\vec{u}\| \sin\theta = \|\vec{v} \times \vec{u}\|$.

□

Example 2.40. Find the area of the parallelogram with corners at $(-1, 0), (2, 0), (1, 3), (4, 3)$.

We see that two of the vectors here are $3\vec{i}$ and $2\vec{i} + 3\vec{j}$. Thus we have

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & 0 \\ 2 & 3 & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + (9-0)\vec{k} = 9\vec{k}$$

so $\|\vec{u} \times \vec{v}\| = 9$.

3 Differentiation

Now that we have a basic understanding of multivariable functions, we want to apply calculus to them. Our goal in this section is to define and understand the derivative, which measures the rate at which a function is changing.

3.1 The Partial Derivative

Already during this class, we have often talked about how quickly a function is changing when you change one of the input variables. This is exactly the single-variable calculus derivative and can be defined accordingly.

Definition 3.1. Let f be a function of two variables. Then we define the *partial derivatives at the point* (a, b) by

$$\begin{aligned} \frac{\partial f}{\partial x}(a,b) &= \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h} = f_x(a,b) \\ \frac{\partial f}{\partial y} &= \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h} = f_y(a,b). \end{aligned}$$

If we allow (a, b) to vary, we get functions $f_x(x, y)$ and $f_y(x, y)$.

We will sometimes write $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. If we want to represent these derivatives evaluated at a point, we will write $\frac{\partial z}{\partial x}\Big|_{(a,b)}$ and $\frac{\partial z}{\partial y}\Big|_{(a,b)}$.

Remark 3.2. This isn't just analogous to the single-variable calculus derivative; it is exactly identical. If we have a function $f : \mathbb{R}^2 \to \mathbb{R}$ and we hold the second variable fixed at $y = b$, then we get a single-variable function defined by $f_b(x) = f(x, b)$. Then $f_x(a, b) = f_b'(a)$ is just the single-variable derivative of this single-variable function.

The interesting part here is not that we can define the partial derivatives, which are basically old news. The interesting thing is that we can get the answers to genuinely multi-varaible questions out of these essentially single-variable tools.

Example 3.3. Suppose a differentiable function $f(x, y)$ has the following values:

$y \setminus x$	0	1	2	3	4	5
0	120	135	155	160	160	150
1	125	128	135	160	175	160
2	100	110	120	145	190	170
3	85	90	110	135	155	180

Then we can estimate the partial derivatives off the chart. For instance, we can estimate that $f_x(3,1)$ is about 20: since $f(4,1) - f(3,1) = 15$ and $f(3,1) - f(2,1) = 25$. Similarly, we can estimate $f_y(3,1) \approx -7.5$ since $f(3,1) - f(3,0) = 0$ and $f(3,2) - f(3,1) = -15$.

One way to understand partial derivatives is to think about the units of the function. For instance, in your homework (problem 12.3.26) you looked at a function $H(x,t)$ that took position and time as inputs, and had temperature as an output. Then $H_x(x,t)$ has units of degrees per meter—how quickly temperature changes when you move a foot away. And $H_t(x,t)$ has units of degrees per minute—how quickly temperature changes over time.

Partial derivatives are easy and quite boring to calculate. Since we're looking at $f(x,y)$ as a function of a single variable, while holding the other constant, we can pretend it's simply a single-variable function, and treat the other variable like a constant.

Example 3.4. Let $f(x,y) = x^2 + y^2$. Then $f_x(x,y) = 2x$ and $f_y(x,y) = 2y$.

Let $g(x,y) = \sin(xy)$. Then $g_x(x,y) = \cos(xy) \cdot y$ and $g_y(x,y) = \cos(xy) \cdot x$.

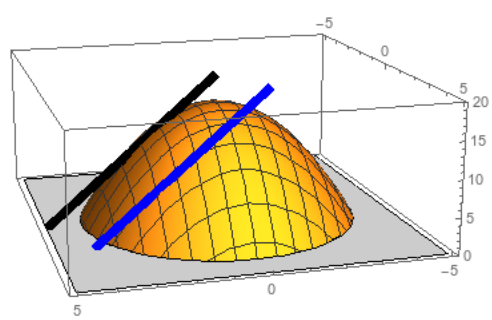
Let $h(x,y) = \frac{x^2}{y^3-3y}$. Then $h_x(x,y) = \frac{2x}{y^3-3y}$ and $h_y(x,y) = -\frac{x^2(3y^2-3)}{(y^3-3y)^2}$.

We can graphically understand partial derivatives by thinking about the cross-section.

Example 3.5. Let $f(x,y) = 16 - x^2 - y^2$. Then $f_x(x,y) = -2x$. Thus $f_x(2,0) = -4$, and the cross-section at 0 is $f(x,0) = 16 - x^2$ and has tangent line $z - 12 = -4(x-2)$.

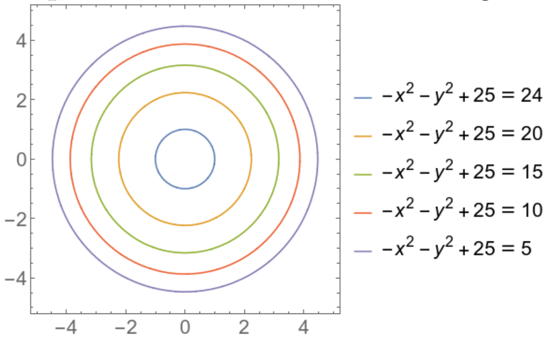
Similarly, if we look at the point $(2,2)$, we see that the cross-section is $f(x,2) = 12 - x^2$ and the derivative is $f_x(2,2) = -4$, so the tangent line is $z - 8 = -4(x-2)$.

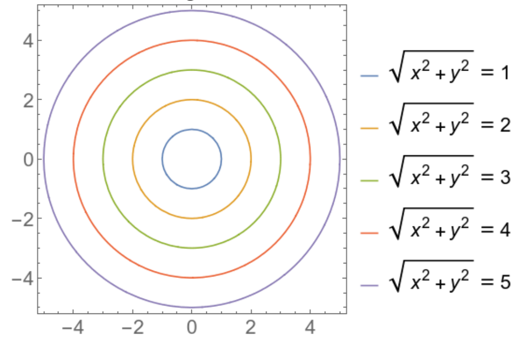
Notice that the slopes of both lines are the same, since here $f_x(x,y)$ doesn't depend on y.



In section 1.2.4 we talked about reading contour diagrams and thinking about in which directions the function was changing. We can interpret this in terms of partial derivatives.

Example 3.6. Recall the contour diagrams we saw in figure 1.9:

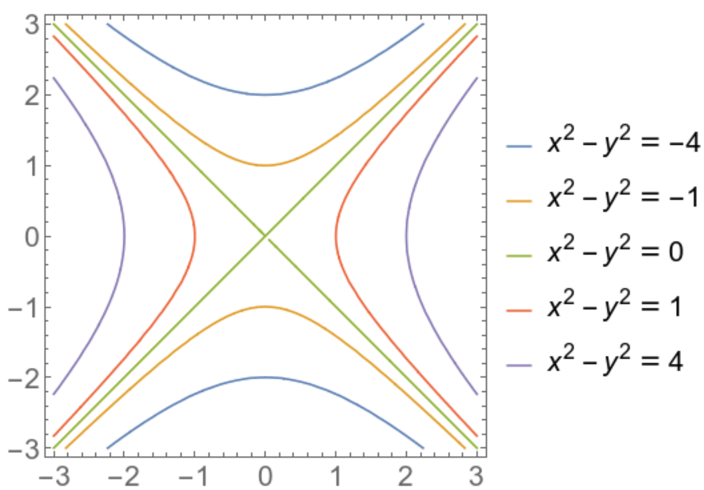




We can ask questions like $f_x(1,0)$ and $g_x(1,0)$. Looking at the graph, we see that $f_x(1,0) \approx -4$ since it changes from 24 to 20 between $(1,0)$ and $(2,0)$. We can see that $f_y(1,0)$ is slightly smaller, since going from $(1,0)$ to $(1,1)$ doesn't quite get us from 24 to 20.

Similarly, $g_x(-2,0)$ is about -1, since $g(-3,0) = 3, g(-2,0) = 2$, and $g(-1,0) = 1$. $g_y(-2,0)$ is positive but less than 1.

Example 3.7. In the picture below, is $f_x(0,2)$ positive, negative, or zero? Is $f_y(0,2)$ positive, negative, or zero?



$f_x(0,2)$ is zero, since the curve is flat there and moving to the left or right shouldn't increase or decrease the output.

$f_y(0,2)$ is negative since the output gets lower as we go up away from the origin.

We can also define the partial derivatives in three (or more) dimensions; the only thing that changes is that the picture becomes more difficult to draw.

Example 3.8. Let $f(x,y,z) = x^2 + xyz + y/z$. Then we have

$$f_x(x,y,z) = 2x + yz$$
$$f_y(x,y,z) = zy + 1/z$$
$$f_z(x,y,z) = xy - y/z^2.$$

3.2 Local Linear Approximation

In many ways, the most important application of the derivative is the ability to approximate a function with a linear function. The basic idea is the same as the idea from single-variable calculus. If you zoom in enough on a 1-variable function, it will loook mostly like a line; if you zoom in on a 2-variable function, it will look like a plane.

Definition 3.9. Roughly speaking, the *tangent plane* to a surface at the point (x,y,z) is a plane that passes through the point (x,y,z), and touches the surface only at that point.

Proposition 3.10. *If $f(x,y)$ is differentiable at the point (a,b), then the equation of the tangent plane to $z = f(x,y)$ at the point (a,b) is*

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

From the equation form, we see that this plane must pass through the point $(a,b,f(a,b))$. Further, the slope in the x direction is $f_x(a,b)$, which is the rate at which f is changing when you change x. Similarly, $f_y(a,b)$ is the slope in the y direction.

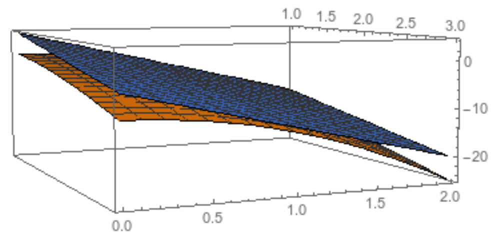
Example 3.11. Let's find the tangent plane to the function $f(x,y) = -x^2 - 4y^2$ at the point $(2,1,-8)$.

We compute

$$f_x(x,y) = -2x \qquad\qquad f_x(2,1) = -4$$
$$f_y(x,y) = -8y \qquad\qquad f_y(2,1) = -8.$$

Since $f(2,1) = -8$, the equation for the tangent plane is

$$z = -8 - 4(x-2) - 8(y-1)$$



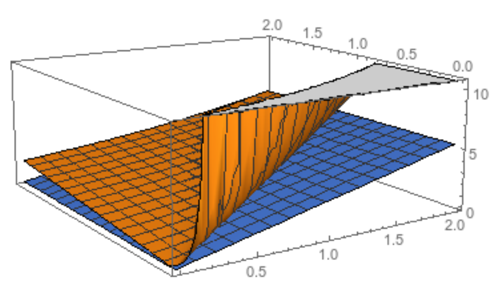
Example 3.12. Let's find the tangent plane to the function $g(x, y) = ye^{x/y}$ at the point $(1, 1)$.

We compute

$$\begin{aligned}
g_x(x, y) &= ye^{x/y}\frac{1}{y} = e^{x/y} \\
g_x(1, 1) &= e \\
g_y(x, y) &= e^{x/y} + ye^{x/y}\frac{-x}{y^2} = e^{x/y} - \frac{x}{y}e^{x/y} \\
g_y(1, 1) &= e - e = 0.
\end{aligned}$$

Since $g(1, 1) = e$, the equation for the tangent plane is

$$z = e + e(x - 1).$$



As with linear functions in single-variable calculus, we can use the tangent plane to approximate the values of a funtion.

Example 3.13. Let's estimate $g(1.1, 1)$.

We know that

$$g(x,y) \approx e + e(x-1)$$
$$g(1.1,1) \approx e + e(1.1-1) = e + .1e = 1.1e.$$

Using Mathematica, we compute that $g(1.1,1) \approx 3.00417$, and $1.1e \approx 2.99011$, so this is pretty good.

Definition 3.14. The *tangent plane approximation* to a function $f(x,y)$ near the point (a,b) is given by

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

The *linear approximation* to a function $f(x,y,z)$ near the point (a,b,c) is given by

$$f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c).$$

Sometimes this is phrased in terms of the differential.

Definition 3.15. The *differential* df of a function f at a point (a,b) is a linear function in the variables dx and dy, given by

$$df = f_x(a,b)dx + f_y(a,b)dy.$$

We will sometimes write $df = f_x dx + f_y dy$.

We can interpret the differential as being, for each point (a,b), a linear function that takes in a change in the x and y coordinates and outputs a change in the z coordinate. Thus

$$f(a+dx, b+dy) \approx f(a,b) + df(dx,dy) = f_x(a,b)dx + f_y(a,b)dy.$$

3.3 Gradients and directional derivatives

In the previous sections we used the partial derivatives to tell us how $f(x,y)$ will change as we change the input variables x and y. We'd like to generalize this further, and see what happens when we change the input in an arbitrary direction.

Definition 3.16. Let $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ be a unit vector. Then we define the *directional derivative* of f in the direction $\vec{u}$ at the point (a,b,c) to be

$$f_{\vec{u}}(a,b,c) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2, c+hu_3) - f(a,b)}{h}$$

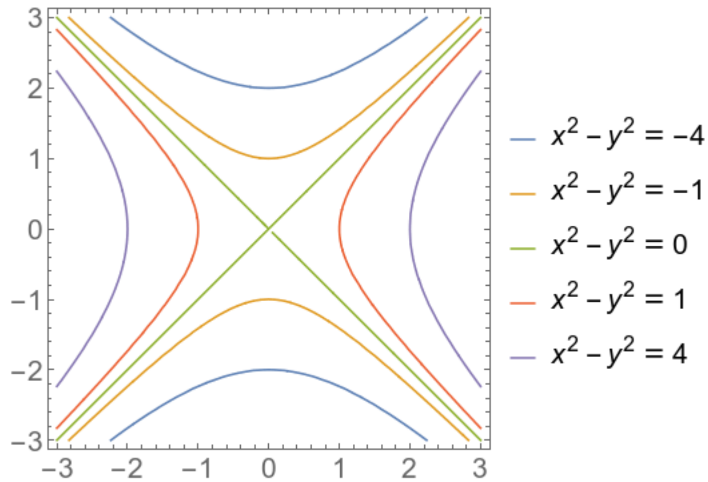
to be the rate of change of f in the direction $\vec{u}$.

If $\vec{v}$ is a non-zero non-unit vector, then we say the directional derivative in the direction of $\vec{v}$ is the directional derivative in the direction of $\frac{\vec{v}}{\|\vec{v}\|}$.

Conceptually, here we're seeing what happens if we change the input in the direction $\vec{u}$ with a change of size h, and then letting the size of the change go to zero.

Remark 3.17. If $\vec{u} = \vec{i}$, then $f_{\vec{u}} = f_x$. Similarly $f_{\vec{j}} = f_y$ and $f_{\vec{k}} = f_z$.

Example 3.18. Let's look at some of our contour plot from section 3.1 again.



We can ask for directional derivatives at a point. If we look at the point $(0, 0)$, we can see the derivative in the $\vec{i}$ direction is positive, and the derivative in the $\vec{j}$ direction is negative; these are just the partial derivatives we've already discussed.

But we can also see that the derivative in the $\vec{i} + \vec{j}$ direction is zero, since it follows directly along the contour.

Now think about the point $(1, -3)$. Is the directional derivative in the $\vec{i} + \vec{j}$ direction positive or negative? It should be positive, again, since we're climing up past the -4 contour towards the -1 contour.

What direction should we go to have a zero directional derivative? It's hard to be exact, but it looks like it should be down-right, and more right than down (following roughly parallel to the blue contour). In fact, we can compute that the exact direction is $3\vec{i} - \vec{j}$; we will see how to compute this later in this section.

We can compute these directional derivatives directly from the definition.

Example 3.19. Let $f(x) = x^2 - y^2$ (the function whose contour plot is in example 3.18). Let's compute the directional derivative in the $\vec{i} + \vec{j}$ direction at the point $(1, -3)$. Our unit vector in that direction is $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$, and we compute

$$\begin{aligned}
f_{\vec{u}}(1,-3) &= \lim_{h\to 0} \frac{f(1+h/\sqrt{2}, -3+h/\sqrt{2}) - f(1,-3)}{h} \\
&= \lim_{h\to 0} \frac{(1+h/\sqrt{2})^2 - (-3+h/\sqrt{2})^2 - (1^2 - (-3)^2)}{h} \\
&= \lim_{h\to 0} \frac{1+\sqrt{2}h + h^2/2 - (9 - 3\sqrt{2}h + h^2/2) - (-8)}{h} \\
&= \lim_{h\to 0} \frac{4\sqrt{2}h}{h} = \lim_{h\to 0} 4\sqrt{2} = 4\sqrt{2}.
\end{aligned}$$

Computing the directional derivative directly from the limit definition is completely possible, but it's tedious. Just as we found easy ways to compute the single-variable derivative, we would like easy ways to compute the directional derivative of a multivariable function.

Fortunately, the partial derivatives give us enough information to do this. By local linearity, we see that

$$\begin{aligned}
f(a+hu_1, b+hu_2) &\approx f(a,b) + hu_1 f_x(a,b) + hu_2 f_y(a,b) \\
\frac{f(a+hu_1, b+hu_2) - f(a,b)}{h} &\approx \frac{hu_1 f_x(a,b) + hu_2 f_y(a,b)}{h} = u_1 f_x(a,b) + u_2 f_y(a,b).
\end{aligned}$$

Since this approximation should get increasingly good as h gets small, we conclude that

$$f_{\vec{u}}(a,b) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h} = u_1 f_x(a,b) + u_2 f_y(a,b).$$

Example 3.20. Let's work out our previous example this way. If $f(x,y) = x^2 - y^2$, we see that $f_x(x,y) = 2x$ and $f_y(x,y) = -2y$. Thus $f_x(1,-3) = 2$ and $f_y(1,-3) = 6$. Then we have

$$f_{\vec{u}}(1,-3) = \frac{1}{\sqrt{2}} \cdot 2 + \frac{1}{\sqrt{2}} \cdot 6 = \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

as we got before.

In this computation, you may notice that we have something that looks like a dot product of $\vec{u}$ with a vector containing the partial derivatives. This leads us to define an object that we will use in almost all of our derivative calculations in the future.

Definition 3.21. If $f(x,y)$ is differentiable at (a,b), then the *gradient vector* of f at (a,b) is

$$\operatorname{grad} f(a,b) = \nabla f(a,b) = f_x(a,b)\vec{i} + f_y(a,b)\vec{j}.$$

Similarly, if $f(x,y,z)$ is differentiable at (a,b,c), then the gradient vector is

$$\operatorname{grad} f(a,b,c) = \nabla f(a,b,c) = f_x(a,b,c)\vec{i} + f_y(a,b,c)\vec{j} + f_z(a,b,c)\vec{k}.$$

Remark 3.22. We sometimes say that

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}.$$

This is just another way of writing the same definition, but is really notationally convenient.

Proposition 3.23. *If f is differentiable at (a,b,c) and $\vec{u}$ is a unit vector, then*

$$f_{\vec{u}}(a,b,c) = \nabla f(a,b) \cdot \vec{u}.$$

Example 3.24. Let $f(x,y) = xy - \sin(x)$. Then the gradient is

$$\nabla f(x,y) = (y - \cos(x))\vec{i} + x\vec{j}$$

and the gradient at the point $(\pi, 1)$ is

$$\nabla f(\pi, 1) = 2\vec{i} + \pi\vec{j}.$$

The directional derivative in the direction $3/5\vec{i} + 4/5\vec{j}$ is

$$(2\vec{i} + \pi\vec{j}) \cdot (3/5\vec{i} + 4/5\vec{j}) = \frac{6 + 4\pi}{5}.$$

The gradient tells us basically everything we want to know about the derivative of the function f; in many ways it "is" the derivative. (From a linear algebra perspective, ∇f is the matrix corresponding to the local linearization of f).

Proposition 3.25. *If f is differentiable at (a,b,c) and $\nabla f(a,b,c) \neq \vec{0}$, then:*

- *$\nabla f(a,b,c)$ is in the direction of maximum increase for f.*
- *$\|\nabla f(a,b,c)\|$ is the maximum rate of increase of f in any direction.*
- *$\nabla f(a,b,c)$ is perpendicular to the level sets of f.*

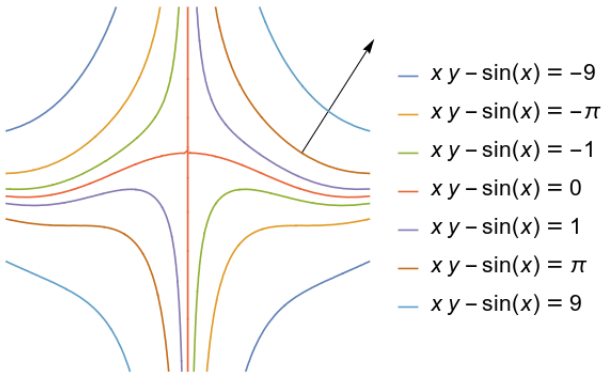
Proof. The rate of increase in the direction of a unit vector $\vec{u}$ is

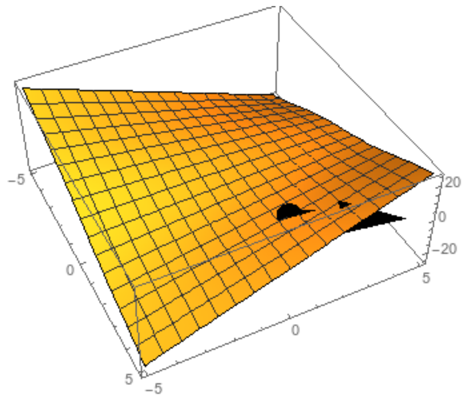
$$\nabla f(a,b,c)\dot{\vec{u}} = \|\nabla f(a,b,c)\| \cdot \|\vec{u}\| \cos\theta = \|\nabla f(a,b,c)\| \cos\theta.$$

This is maximized when $\theta = 0$, which is when $\nabla f(a,b,c)$ and $\vec{u}$ point in the same direction; the maximum value is $\|\nabla f(a,b,c)\|$.

$\nabla f(a,b,c)$ is the normal vector to the tangent plane (or line) at (a,b,c), and thus is perpendicular to the tangent plane. Thus it is perpendicular to the level set. $\square$

Example 3.26. We can look at the contour diagram and the graph for the function $f(x, y) = xy - \sin(x)$ from example 3.24.



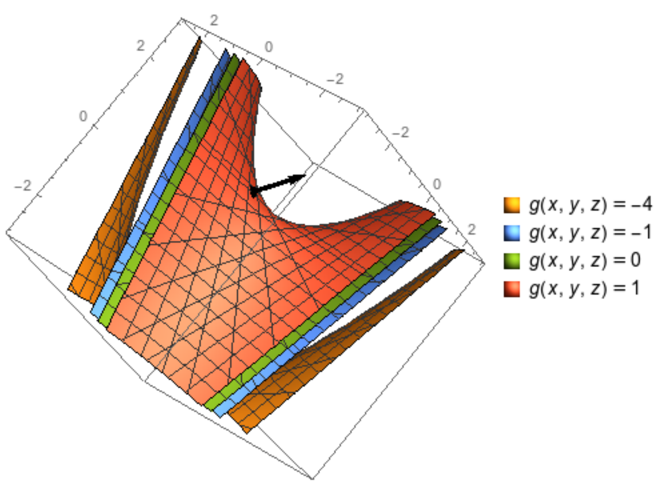


We see in the contour diagram that the gradient vector is perpendicular to the contour, and is in the direction of greatest increase. We can see the latter again in the three-dimensional graph—but this is much harder to read and see what's happening.

Example 3.27. Let's do a three-variable example next. Let $g(x, y, z) = xy + z$. Then

$$\nabla g(x, y, z) = y\vec{i} + x\vec{j} + 1\vec{k}.$$

At the point $(-1, 0, 1)$, we have $\nabla g(x, y, z) = -\vec{j} + \vec{k}$. Thus the direction of greatest increase is $-\vec{j} + \vec{k}$ and the magnitude of the increase in that direction is $\sqrt{2}$.



What if we want the directional derivative in the direction of, say $\vec{v} = 2\vec{i} + \vec{k}$? Then we have

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{2}{\sqrt{5}}\vec{i} + \frac{1}{\sqrt{5}}\vec{k}$$

$$f_{\vec{u}}(-1, 0, 1) = (-\vec{j} + \vec{k}) \cdot \vec{u} = 0 \cdot \frac{2}{\sqrt{5}} - 1 \cdot 0 + \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}}.$$

3.4 The Chain Rule

We'd like an analogue of the single-variable chain rule for multivariable functions. In the single-variable case, we ask how much f changes when you change x, and then how much g changes when you change $f(x)$, and multiply those together: $\frac{d}{dx}g(f(x)) = \frac{dg}{dx}(f(x)) \cdot \frac{df}{dx}(x)$.

The intuition in the multivariable case is basically the same; we track what effect changing each input has, and multiply them through. The expressions are more complicated pretty purely because there are more levers we can push on to change things.

To build some intuition, we'll start with the case where our composite isn't *really* a multivariable function: f depends on two variables, but each of those variables depends only on some variable t. This corresponds to, say, the force acting on a particle over time, when the force depends on position in space and the position in space depends on time.

Proposition 3.28 (Parametrized Chain Rule)**.** *If f, g, h are differentiable, and $x = g(t)$ and $y = h(t)$, then*

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Conceptually, what's happening here is that we look at every way that f can change, and then see how t affects each of those factors; then we add all the separate changes together. (This is making some implicit assumption that things are almost linear—but every time we use the derivative, we're making that assumption).

Sketch. We know that $\Delta f \approx \frac{\partial f}{\partial x} \cdot \Delta x + \frac{\partial f}{\partial y} \cdot \Delta y$. But further we know that $\Delta x \approx \frac{dx}{dt} \cdot \Delta t$ and $\Delta y \approx \frac{dy}{dt} \cdot \Delta t$. Putting this together gives us

$$\begin{aligned} \Delta f &\approx \frac{\partial f}{\partial x}\frac{dx}{dt}\Delta t + \frac{\partial f}{\partial y}\frac{dy}{dt}\Delta t \\ \frac{\Delta f}{\Delta t} &\approx \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \end{aligned}$$

and taking the limit as t goes to zero gives us what we want. $\square$

Example 3.29. Suppose $z = y\cos(x)$, where $x = t^2$ and $y = t^3$. Then

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (-y\sin(x)) \cdot 2t + \cos(y) \cdot 3t^2 \\ &= -t^3\sin(t^2) \cdot 2t + \cos(t^3) \cdot 3t^2. \end{aligned}$$

We can generalize this sort of chain rule behavior to chaining together functions of many variables. In general, we have

$$\frac{\partial z}{\partial t} = \sum_{x_i} \frac{\partial z}{\partial x_i} \cdot \frac{\partial x_i}{\partial t}.$$

That is, for each variable that z depends on, we multiply together the way z depends on the variable and the way the variable depends on t, and then add these all together to get the total change.

Example 3.30. Let $f(x,y) = x^2 y$ where $x = 4u + v$ and $y = u^2 - v^2$. Compute $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$.

$$\begin{aligned}
\frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 2xy \cdot 4 + x^2 \cdot 2u \\
&= 2(4u+v)(u^2 - v^2)4 + (4u+v)^2 2u \\
\frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = 2xy \cdot 1 + x^2(-2v) \\
&= 2(4u+v)(u^2 - v^2) + (4u+v)^2(-2v).
\end{aligned}$$

Example 3.31. Suppose we have a function f that can be expressed as a function of x and y, or of u and v, where $x = u + v$ and $y = u - v$. (This is called a change of basis). We can express the partial derivatives in terms of each other.

We have

$$\begin{aligned}
\frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial f}{\partial x} \cdot 1 + \frac{\partial f}{\partial y} \cdot 1 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \\
\frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial f}{\partial x} \cdot 1 + \frac{\partial f}{\partial y} \cdot (-1) \\
&= \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}.
\end{aligned}$$

If we want to go the opposite way, and express $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in terms of $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$, then we have two options. One is to observe that $u = \frac{x+y}{2}$ and $v = \frac{x-y}{2}$, and then use the chain rule again:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = \frac{1}{2}\frac{\partial f}{\partial u} + \frac{1}{2}\frac{\partial f}{\partial v} \\
\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial u} - \frac{1}{2}\frac{\partial f}{\partial v}.
\end{aligned}$$

Alternatively, we could have taken the expressions we already had and rearranged them. We knew that

$$\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} = 2\frac{\partial f}{\partial x}$$
$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\frac{\partial f}{\partial y}$$

and dividing by 2 gives us the same answer we got before.

3.5 Second Partials

So far we've spoken explicitly only about the first-order derivatives of f. But each derivative gives us a new function, which we can also take the derivatives of. In single variable calculus this gives us "the" second derivative. In multivariable calculus, just as there is more than one first derivative, there is more than one second derivative.

Definition 3.32. We define the *second-order partial derivatives* of $f(x, y)$ to be

$$\frac{\partial^2 z}{\partial^2 x} = f_{xx} = (f_x)_x \qquad \frac{\partial^2 z}{\partial x \partial y} = f_{yx} = (f_y)_x$$
$$\frac{\partial^2 z}{\partial y \partial x} = f_{xy} = (f_x)_y \qquad \frac{\partial^2 z}{\partial^2 y} = f_{yy} = (f_y)_y$$

Example 3.33. Let $f(x, y) = xy^2 + 3x^2e^y$. Then

$$f_x(x, y) = y^2 + 6xe^y \qquad f_y(x, y) = 2xy + 3x^2e^y$$

so we compute

$$f_{xx}(x, y) = 6e^y \qquad f_{yx}(x, y) = 2y + 6xe^y$$
$$f_{xy}(x, y) = 2y + 6xe^y \qquad f_{yy}(x, y) = 2x + 3x^2e^y.$$

You may have noticed a repetition here. Though there are four distinct mixed partials to compute, we only got three separate answers. This isn't an accident.

Theorem 3.34. *If f_{xy} and f_{yx} are continuous at the point (a, b), and (a, b) is an interior point of their domain, then*

$$f_{xy}(a, b) = f_{yx}(a, b).$$

These second-order partials measure how quickly the derivatives—the first partials—change when we change our input. This is similar to your homework problem 14.1.24, which asked how the partial derivatives changed as you moved from point A to point B.

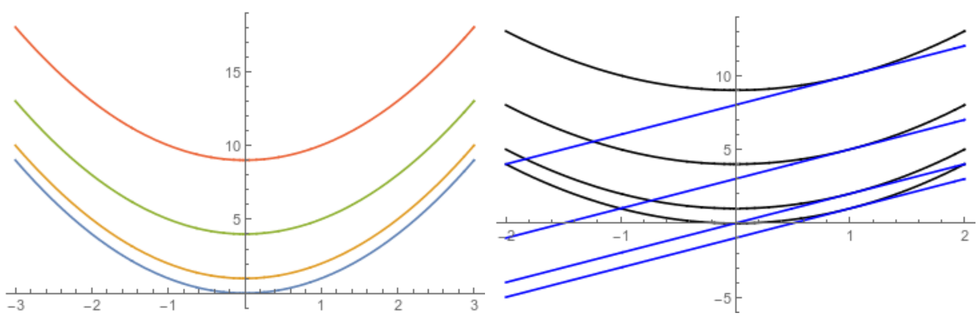
For example, if f_{xx} is positive, that means that the function gets steeper in the x direction as you increase x. If f_{xy} is positive, that means the function gets steeper in the x diretion as you increase y.

Example 3.35. Consider the function $f(x, y) = x^2 + y^2$. We see that

$$f_{xx}(x, y) = 2 \qquad f_{xy}(x, y) = 0 \qquad f_{yy}(x, y) = 2.$$

What does this tell us? Well, at any point, moving one unit in the x direction increases the x slope by about two; and similarly, moving one unit in the y direction increases the y slope by about two.

But moving in the y direction doesn't affect the x slope at all, and vice versa. Geometrically, this is because all the cross sections are identical parabolas at different heights: their levels will be different, but their slopes will all be the same at the same x value. So changing y doesn't change the x slope at all.



We can use these second partial derivatives to improve our approximations. In section 3.2 we talked about linear approximation, which the linear function that best approximates our function near a given point. With second partials, we can construct the *second-degree Taylor polynomial* or *quadratic approximation*.

Definition 3.36. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined near (a, b). The *Taylor polynomial of degree 1* for f near (a, b) is

$$T_1(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

The *Taylor polynomial of degree 2* is

$$\begin{aligned} T_2(x, y) = f(a, b) &+ f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ &+ \frac{f_{xx}(a, b)}{2}(x - a)^2 + f_{xy}(0, 0)(x - a)(y - b) + \frac{f_{yy}(a, b)}{2}(y - b)^2. \end{aligned}$$

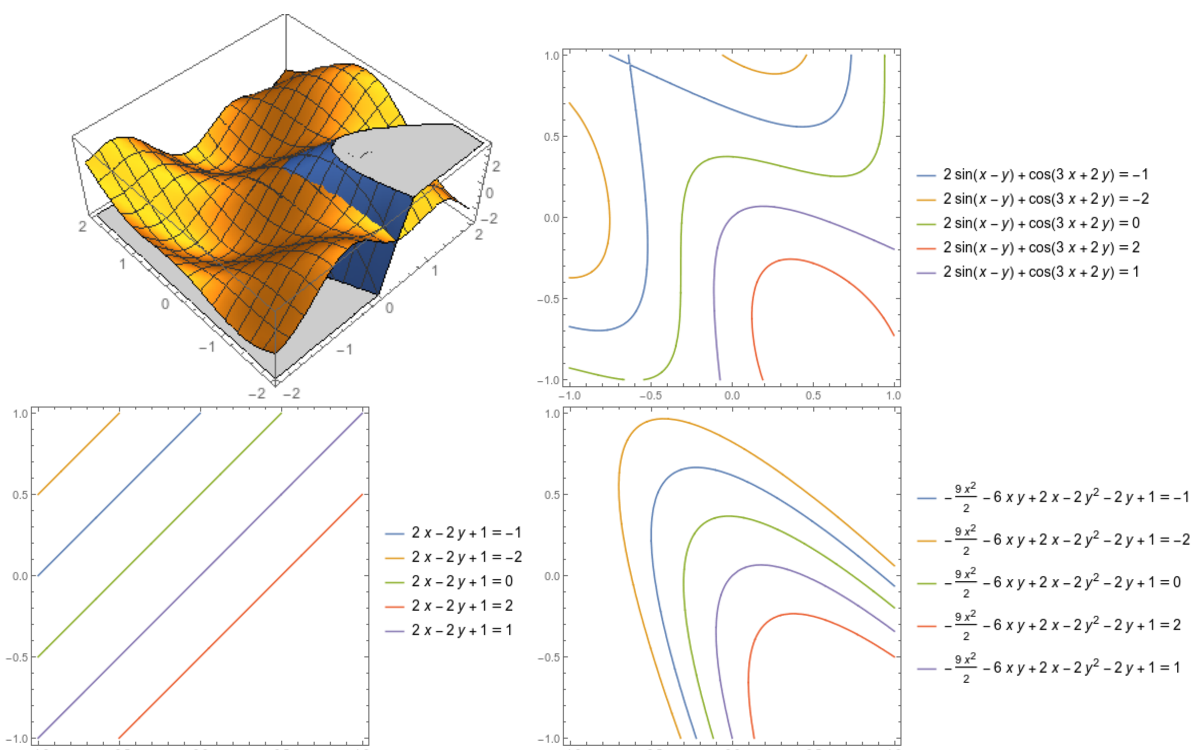
These approximations are used quite often in physics and in any other sort of numeric approximation work. It's possible to go to third-order and higher, and this works exactly like you'd expect; but third-order approximations are rarely actually useful. If the quadratic approximation isn't good enough, you generally want to just use a better tool instead.

Example 3.37. Let's find a quadratic approximation to $\cos(3x + 2y) + 2\sin(x - y)$ near $(0, 0)$.

$$\begin{aligned}
f(x,y) &= \cos(3x+2y) + 2\sin(x-y) & f(0,0) &= 1 \\
f_x(x,y) &= -3\sin(3x+2y) + 2\cos(x-y) & f_x(0,0) &= 2 \\
f_y(x,y) &= -2\sin(3x+2y) - 2\cos(x-y) & f_y(0,0) &= -2 \\
f_{xx}(x,y) &= -9\cos(3x+2y) + 2\sin(x-y) & f_{xx}(0,0) &= -9 \\
f_{xy}(x,y) &= -6\cos(3x+2y) + 2\sin(x-y) & f_{xy}(0,0) &= -6 \\
f_{yy}(x,y) &= -4\cos(3x+2y) - 2\sin(x-y) & f_{yy}(0,0) &= -4
\end{aligned}$$

so the quadratic approximation is

$$T_2(x, y) = 1 + 2x - 2y - 9x^2/2 - 6xy - 2y^2.$$



Suppose we want to find $\cos(.3 - .2) + 2\sin(.1 + .1)$. Then we have

$$f(.1, -.1) \approx T_2(.1, -.1) = 1 + .2 + .2 - .09/2 + .06 - .02 = 1.395.$$

Plugging in, the true answer is ≈ 1.39234, so this is pretty good.

Example 3.38. Let's find a quadratic approximation to e^{xy} near $(0, 2)$.

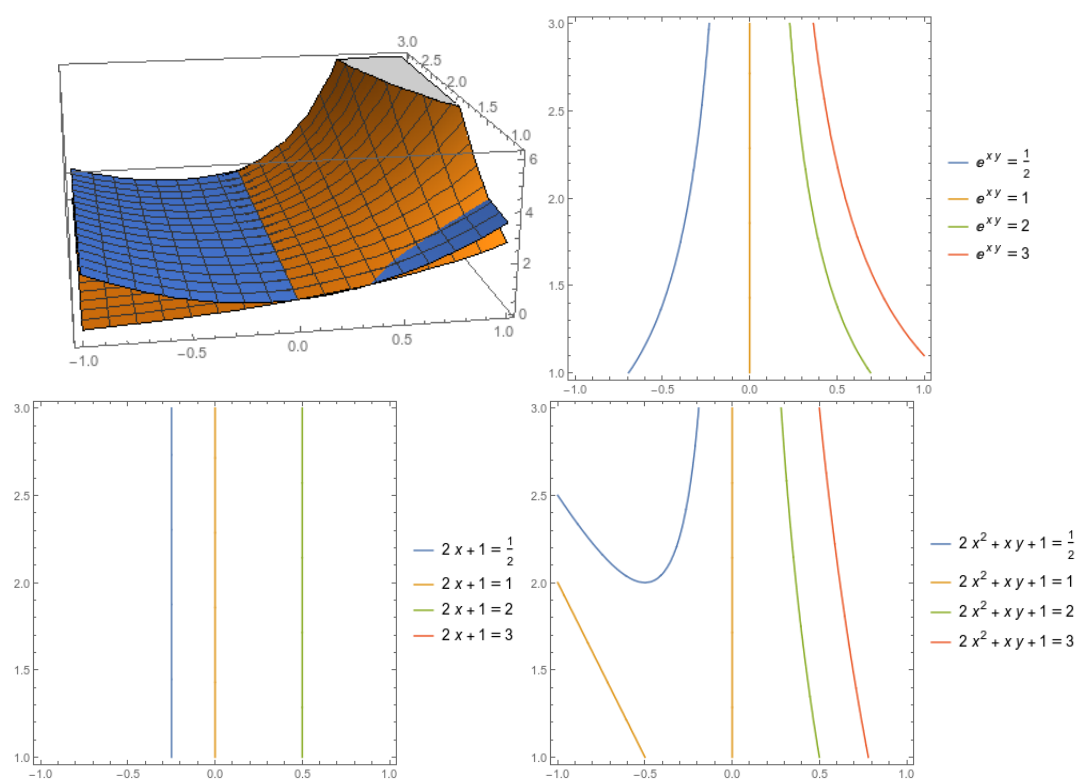
We compute

$$\begin{aligned}
f(x,y) &= e^{xy} & f(0,2) &= 1 \\
f_x(x,y) &= ye^{xy} & f_x(0,2) &= 2 \\
f_y(x,y) &= xe^{xy} & f_y(0,2) &= 0 \\
f_{xx}(x,y) &= y^2e^{xy} & f_{xx}(0,2) &= 4 \\
f_{xy}(x,y) &= e^{xy} + xye^{xy} & f_{xy}(0,2) &= 1 \\
f_{yy}(x,y) &= x^2e^{xy} & f_{yy}(0,2) &= 0
\end{aligned}$$

Thus we can compute the Taylor polynomial:

$$\begin{aligned}
T_2(x,y) &= 1 + 2x + 0(y-2) + 4x^2/2 + 1 \cdot x(y-2) + 0(y-2)^2/2 \\
&= 1 + 2x + 2x^2 + x(y-2) = 1 + 4x^2 + xy
\end{aligned}$$

(We can multiply it out like in that last step; we generally shouldn't).



We can see that the linear approximation is still trying but not quite there.

We can also estimate, say, $e^{(-.1)\cdot 2.2} = e^{-.22}$. We have

$$e^{-.22} = f(-.1, 2.2) \approx T_2(-.1, 2.2) = 1 + .02 - .22 = .80.$$

Alternatively, we could write

$$e^{-.22} = f(-.1, 2.2) \approx T_2(-.1, 2.2) = 1 - .2 + .02 - .1(.2) = .8.$$

The true answer is about .8025.

4 Optimization

There are two major applications of derivatives. The first, which we explored in sections 3.2 and 3.5, is to approximate functions that are hard or annoying to compute. The other is to attempt to find optimal values of functions.

The case is basically similar to the single-variable case, but as usual some extra wrinkles are introduced by having more than one input variable.

4.1 Critical points and Local Extrema

Definition 4.1. We say f has a *local maximum* at the point P_0 if $F(P_0) \geq f(P)$ for all P near P_0.

We say f has a *local minimum* at the point P_0 if $F(P_0) \leq f(P)$ for all P near P_0.

Remark 4.2. Note that we say f "has" an extremum at P. The extreme value is the actual output of f at that point. Thus, we can't say that P "is" a maximum of f.

It's possible to be very precise about what the word "near" means, but in this case we won't really bother. A point is a local maximum if you can draw a small circle around it and it gives the largest value of any point in that circle.

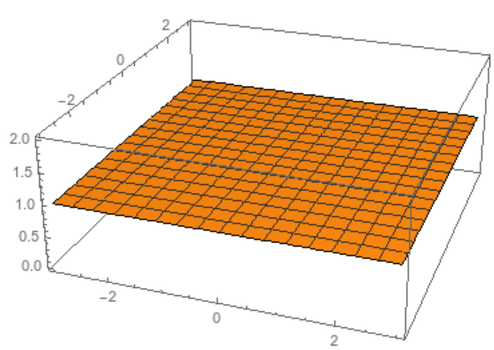
Example 4.3. Let $f(x, y) = 1$. Does this have any global maxima or minima?

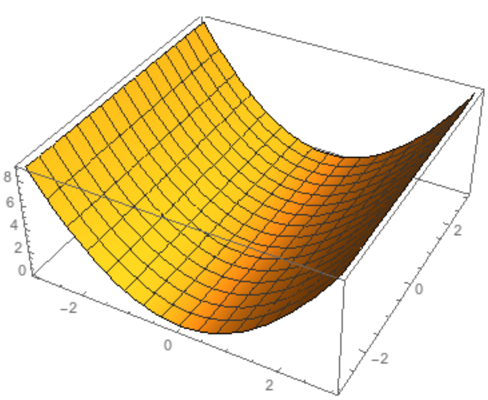
Yes. There is a maximum *and* a minimum at every single point.

This example is actually less silly in the multivariable case than in the single-variable case.

Example 4.4. Let $f(x, y) = x^2$. Does this have any global maxima or minima?

Yes. When $x = 0$ we have a local minimum whose value is zero. Thus there is a minimum at every point on this line.





With a picture (in 2 or 3 dimensions), we can identify the local extrema. And with a sufficiently simple algebraic expression we can figure out what they are. But what can we do when the situation is more complex? We need to use the derivatives.

Theorem 4.5 (Fermat). *If f has a local extremum at P, and $\nabla f(P)$ exists, and P is not on the boundary of the domain of f, then $\nabla f(P) = \vec{0}$.*

Proof. Suppose $\nabla f(P) = \vec{v} \neq \vec{0}$. Then $f_{\vec{v}}(P) > 0$, so $f(P + h\vec{v}) > f(P)$ and so f doesn't have a local maximum at P. Similarly, $f(P - h\vec{v}) < f(P)$ so f doesn't have a local minimum at P. ☐

Definition 4.6. If $\nabla f(P) = \vec{0}$ or ∇f is undefined at P, we say that P is a *critical point* of f.

Thus Fermat's theorem tells us that all (interior) local extrema for f occur at critical points.

Example 4.7. Let $f(x, y) = -\sqrt{x^2 + y^2}$. Then

$$\nabla f(x, y) = \frac{-x}{\sqrt{x^2 + y^2}}\vec{i} + \frac{-y}{\sqrt{x^2 + y^2}}\vec{j}$$

This is actually never equal to zero, since it's undefined at the point $(0, 0)$. But this still makes the origin into a critical point, and indeed we can see that f has a local maximum at the origin.

Example 4.8. Let $f(x, y) = x^2 - 2x + y^2 - 4y + 5$

We compute

$$\nabla f(x, y) = (2x - 2)\vec{i} + (2y - 4)\vec{j}$$

which is $\vec{0}$ precisely when $(x, y) = (1, 2)$. Thus this is the only critical point.

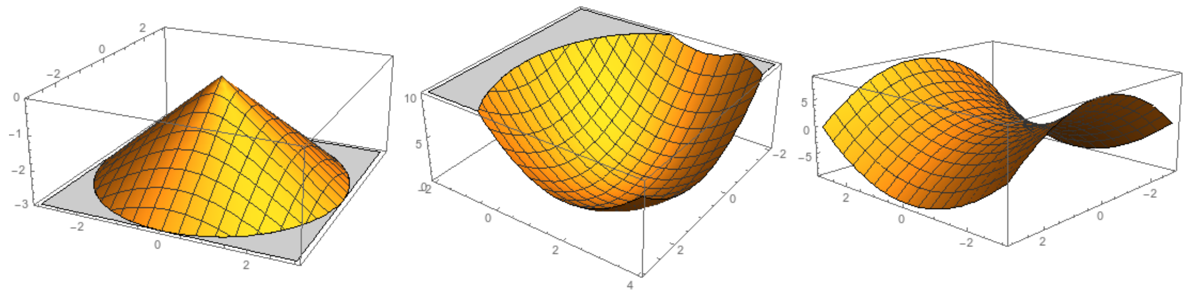
A little algebra tells us that this graph is a paraboloid $(x - 1)^2 + (y - 2)^2$. So there is a minimum at $(1, 2)$ with value 0.

Example 4.9. Let $f(x, y) = x^2 - y^2$. Then

$$\nabla f(x, y) = 2x\vec{i} - 2y\vec{j}$$

is zero when $x = y = 0$. Thus there is a single critical point at $(0, 0)$.

However, from the graph we can see that this is neither a maximum nor a minimum. In fact, it's a minimum in the x direction, and a maximum in the y direction. We call points like this "saddle points".

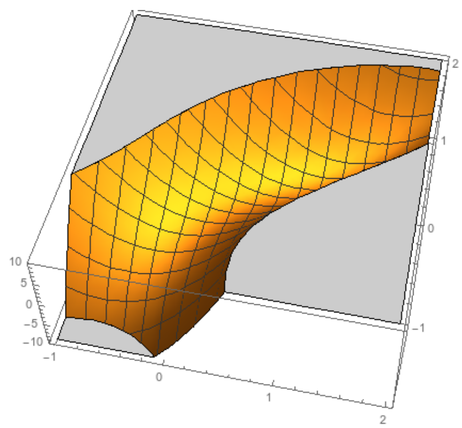


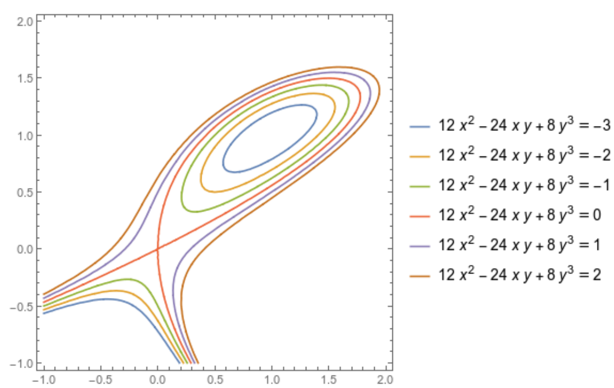
Example 4.10. Let $f(x,y) = 8y^3 + 12x^2 - 24xy$. We compute

$$\nabla f(x,y) = (24x - 24y)\vec{i} + (24y^2 - 24x)\vec{j}.$$

This is zero when $24x = 24y$ and $24y^2 = 24x$, which implies that $x = y$ and $x = y^2$, which gives us either $x = y = 0$ or $x = y = 1$. So there are two critical points, at $(0,0)$ and $(1,1)$.

From looking at the graph, we can see that there is a saddle point at $(0,0)$ and a minimum at $(1,1)$.





This last problem especially is hard to see what's happening without looking at a graph. But the second derivative can tell us what type of extrema we have at critical points.

Proposition 4.11. *Suppose $\nabla f(a,b) = \vec{0}$. Define*

$$D = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2.$$

Then:

- *If $D > 0$ and $f_{xx}(a,b) > 0$, f has a local minimum at (a,b).*
- *If $D > 0$ and $f_{xx}(a,b) < 0$, f has a local maximum at (a,b).*
- *If $D < 0$ f has a saddle point at (a,b).*

Importantly, if $D = 0$ then this proposition doesn't tell us anything and we would need to do something else. We could have a local maximum, a local minimum, a saddle point, or something genuinely weird.

Example 4.12. If $f(x, y) = x^4 + y^4$, then we have

$$\begin{aligned} f_x(x,y) &= 4x^3 & f_y(x,y) &= 4y^3 \\ f_{xx}(x,y) &= 12x^2 & f_{yy}(x,y) &= 12y^2 \\ f_{xy}(x,y) &= 0 & D &= 144x^2y^2. \end{aligned}$$

We see that we have a critical point at $(0, 0)$, but at that point we get $D = 0$, which is unhelpful. But this is clearly a local minimum, since $f(0, 0) = 0$ and $f(x, y) \geq 0$.

If $f(x, y) = -x^4 - y^4$, then we have

$$\begin{aligned} f_x(x,y) &= -4x^3 & f_y(x,y) &= -4y^3 \\ f_{xx}(x,y) &= -12x^2 & f_{yy}(x,y) &= -12y^2 \\ f_{xy}(x,y) &= 0 & D &= 144x^2y^2. \end{aligned}$$

We see that we have a critical point at $(0, 0)$, but at that point we get $D = 0$, which is unhelpful. But this is clearly a local maximum, since $f(0, 0) = 0$ and $f(x, y) \leq 0$.

If $f(x, y) = x^4 - y^4$, then we have

$$\begin{aligned} f_x(x,y) &= 4x^3 & f_y(x,y) &= -4y^3 \\ f_{xx}(x,y) &= 12x^2 & f_{yy}(x,y) &= -12y^2 \\ f_{xy}(x,y) &= 0 & D &= -144x^2y^2. \end{aligned}$$

We see that we have a critical point at $(0, 0)$, but at that point we get $D = 0$, which is again unhelpful. In this case we have a saddle point: we can see that it is a minimum holding y constant, and a maximum holding x constant.

Example 4.13. Let $f(x, y) = x^2/2 + 3y^3 + 9y^2 - 3xy + 9y - 9x$.

We compute

$$\nabla(x, y) = (x - 3y - 9)\vec{i} + (9y^2 + 18y + 9 - 3x)\vec{j}$$

and thus there are critical points when $x = 3y + 9$ and $9y^2 + 18y + 9 = 3x$. Solving this gives

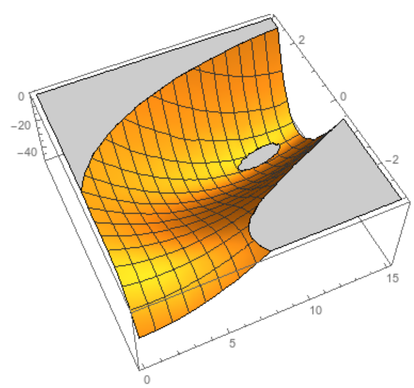
$$\begin{aligned} 9y^2 + 18y + 9 &= 9y + 27 \\ 9y^2 + 9y - 18 &= 0 \\ 9(y + 2)(y - 1) &= 0 \end{aligned}$$

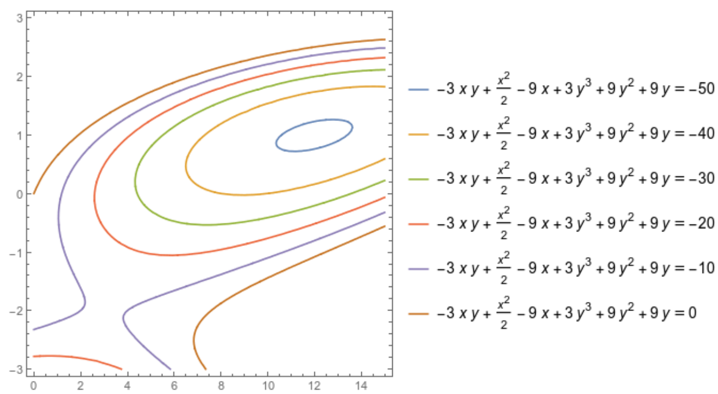
And thus $y = -2$ or $y = 1$. We see that if $y = -2$ then $x = 3$, and if $y = 1$ then $x = 12$, so the critical points are $(3, -2)$ and $(12, 1)$.

For the second derivative test, we have

$$\begin{aligned}
f_{xx}(x, y) &= 1 \\
f_{yy} &= 18y + 18 \\
f_{xy}(x, y) &= -3 \\
D &= (18y + 18) - (-3)^2 = 18y + 9 \\
D(3, -2) &= -27 < 0 \\
D(12, 1) &= 27
\end{aligned}$$

so there is a saddle point at $(3, -2)$ and a minimum at $(12, 1)$.





4.2 Global Extrema and the Extreme Value Theorem

Critical points and the second derivative test let us determine which points are local extrema, but we often also want to know what the largest possible value we can get out of a function is.

Definition 4.14. We say f has a *global maximum* on R at the point P_0 if $F(P_0) \geq f(P)$ for all P in R.

We say f has a *global minimum* on R at the point P_0 if $F(P_0) \leq f(P)$ for all P in R.

Example 4.15. Suppose we are running a factory that produces two different products. The price we can sell each product for depends on the quantity we produce, according to the equations

$$p_1 = 600 - .3q_1 \qquad p_2 = 500 - .2q_2. \tag{1}$$

Our total cost of production is given by

$$C = 16 + 1.2q_1 + 1.5q_2 + .2q_1q_2.$$

We want to know how many of each item to produce to maximize our total profit.

Notice that here we don't really care about the relative extrema; we just want to find the best possible outcome.

First, we need to write our profit as a function of how much of each item we produce. We observe that our revenue is given by

$$R(q_1, q_2) = p_1q_1 + p_2q_2 = 600q_1 - .3q_1^2 + 500q_2 - .2q_2^2.$$

Profit is revenue minus costs, or

$$P(q_1, q_2) = R - C = -16 + 598.8q_1 - .3q_1^2 + 498.5q_2 - .2q_2^2 - .2q_1q_2.$$

Now we have P written as a function of two variables. We want to optimize it on the region $\{(q_1, q_2) : q_1 \geq 0, q_2 \geq 0\}$ since we can't produce negative quantities.

How do we find the largest possible value? In this case, the "physics" (or economics) of the situation tell us that it should occur at a relative maximum, since producing nothing is obviously suboptimal, and we expect our costs to explode as our quantity produced tends to infinity.

(Alternatively, we can notice that our equation is some sort of paraboloid and thus has a unique relative maximum that is also the absolute maximum).

Thus we look for critical points, and compute the partial derivatives.

$$\frac{\partial P}{\partial q_1} = 598.8 - .6q_1 - .2q_2$$
$$\frac{\partial P}{\partial q_2} = 498.5 - .4q_2 - .2q_1$$

and setting these equations equal to zero and solving gives us a critical point at $(q_1, q_2) = (699.1, 896.7)$. Plugging back in to equation (1) gives us prices of $(p_1, p_2) = (390.27, 320.66)$ and we get a total profit of \$432,797 dollars.

We'd like to make sure this is in fact a maximum. We can check the second partials, and we get:

$$\frac{\partial^2 P}{\partial q_1^2} = -.6 \qquad \frac{\partial^2 P}{\partial q_1 \partial q_2} = -.2 \frac{\partial^2 P}{\partial q_2^2} = -.4$$

and thus $D = (-.6)(-.4) - (-.2)^2 = .24 - .04 = .2$. Then $D > 0$ but $\frac{\partial^2 P}{\partial q_1^2} < 0$ and thus we have a local maximum. In fact, since the second derivatives are constant, we see again that we have a paraboloid; we can also infer from this that the function never increases again, so this is the only local maximum and must be a global maximum.

Example 4.16. Suppose a trucker wants to bring 480 cubic meters of gravel to a dump and needs to build a box for transport. Dumping costs \$80 per trip, plus the cost of the box.

The box has height 2m, and costs \$100 per square meter for the ends, \$50 per square meter for the sides, \$200 per square meter for the bottom. What is the optimum box size?

Let's say the box has sides of length x and ends of length y. Then the trucker takes $480/(2xy)$ trips at \$80/trip, for a total cost of $(240 \cdot 80)/(xy)$. The total cost of the box is $400y$ for the ends, $200x$ for the sides, and $200xy$ for the bottom. So total cost is

$$C = 400y + 200x + 200xy + (240 \cdot 80)/(xy) = 200\left(96/(xy) + 2y + x + xy\right).$$

We want to optimize this on the region $\{(x, y) : x > 0, y > 0\}$ since we need a positive-size box.

We can ignore the factor of 200, which doesn't change optimum. Gradient gives

$$C_x = 1 + y - 96/(x^2y) \qquad C_y = 2 + x - 96/(xy^2)$$

Setting equal to zero and solving gives

$$\begin{aligned} 96 &= x^2y + x^2y^2 & \qquad 96 &= 2xy^2 + x^2y^2 \\ x^2y &= 2xy^2 \\ x &= 2y \\ 96 &= 4y^3 + 4y^4 \end{aligned}$$

and the only positive real solution is $y = 2$. Thus the only critical point in the region is $(4, 2)$.The total cost of the transport is \$5600.

We use the second derivative test to make sure this is a minimum. (It certainly ought to be, physically). We see that

$$\begin{aligned} C_{xx} &= 192/(x^3y) & \qquad C_{xy} &= 1 + 96/(x^2y^2) & \qquad C_{yy} &= 192/(xy^3) \\ &= 6 & &= 5/2 & &= 3/2 \end{aligned}$$

and thus

$$D = 9 - 25/4 = 11/4 > 0.$$

Thus $D > 0$ and $C_{xx} > 0$, so this is a local minimum.

In both of these problems, we relied on physical intuition to tell us that a global maximum or minimum should exist. If we don't have such a clear physical setup, how can we tell?

Let's turn the question around and ask how we can *avoid* having a global maximum. One way is for the function to keep increasing infinitely the further we go in some direction. For instance, the function $f(x, y) = x + y$ doesn't have a global maximum on the plane.

Obviously this is only possible if the region is infinite. We say a region is *bounded* if it doesn't extend infinitely in any direction—that is, if we can draw a circle of finite radius around the whole region.

A more subtle way to avoid a maximum is to approach a maximum, and simply not have the point that would give you the maximum. An example here is the function $f(x, y) = x^2+y^2$ on the region $x^2 + y^2 < 1$. You can get any value less than 1, but you cannot get 1—so there is no largest possible value.

This is only possible if the region approaches but doesn't reach some point. We say a region is *closed* if it contains its entire boundary, and thus there are no points approached by the region but not contained in the region.

If a function is continuous, it turns out that these are the only way to avoid having a maximum.

Theorem 4.17 (Extreme Value)**.** *If f is a continuous function on a closed and bounded region R, then f has a global maximum and a global minimum on R.*

Thus if we have a closed and bounded region, and a continuous function, we know it must have a global maximum and a minimum.

In single variable calculus, finding these was easy. We found all the critical points and all the endpoints, plugged them into the function, and then the largest was the global maximum. In the multivariable case things are a bit harder. We still know that the global maximum must appear either at a critical point or a boundary point, but there are infinitely many boundary points so we can't just plug all of them in. Instead we need a technique to find extreme values on the boundary.

4.3 Constrained Optimization and Lagrange Multipliers

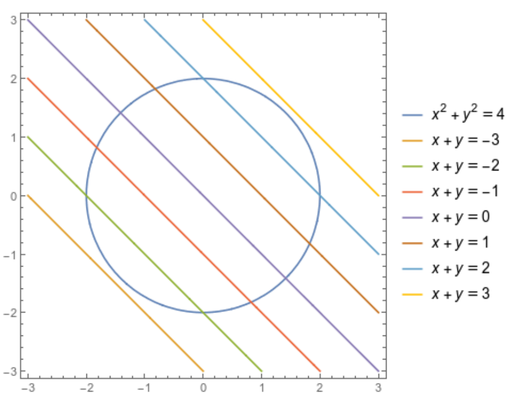
In order to answer this problem, we need to develop techniques for *constrained optimization*: optimization subject to some constraint equation. This will let us find the optimum value of a function on the boundary of its domain; it will also allow us to solve natural problems that ask us to optimize an objective function given a fixed budget, or given some physical limitations on what is possible.

Mathematically, we are looking at the problem: maximize $f(x, y)$ subject to the constraint that $g(x, y) = c$ for some constraint equation g and some constant c.

How do we do this? To find an interior maximum, we look for places where ∇f is zero. But optimizing along a constraint, we don't care if we can increase the value of f by leaving the constraint—we just need the directional derivative to be zero in the direction tangent to $g(x, y) = c$. This is equivalent to asking for ∇f to be perpendicular to that boundary.

But the boundary is just a contour or level set of g, so we know that ∇g is perpendicular to the boundary. So we're really looking for points where ∇f points in the same (or exactly opposite) direction to ∇g. We can impose this condition algebraically by looking for points where $\nabla f = \lambda \nabla g$.

Example 4.18. Let's find the maximum and minimum values of $f(x, y) = x+y$ on $x^2+y^2 = 4$.



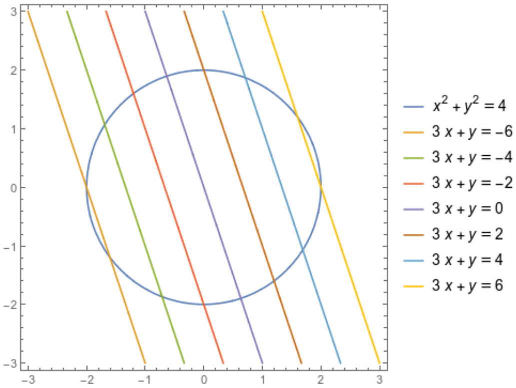
We compute $\nabla f(x, y) = (1, 1)$ and $\nabla g(x, y) = (2x, 2y)$, so we are looking for points where $(1, 1) = \lambda(2x, 2y)$ for some $\lambda \in \mathbb{R}$. This gives us $x = y = 1/(2\lambda)$.

To get specific values, we substitute this back into $x^2 + y^2 = 4$. We get $2x^2 = 4$ so $x^2 = 2$ and thus $x = \pm\sqrt{2}$. We know that $y = x$ so we have two critical points: $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. (We can also see that $\lambda = 1/(2x) = 1/\sqrt{8}$).

Plugging in values, we see that we have a maximum of $2\sqrt{2}$ at $(\sqrt{2}, \sqrt{2})$ and we have a minimum of $-2\sqrt{2}$ at $(-\sqrt{2}, -\sqrt{2})$.

Importantly, notice that this is exactly where you'd expect the maximum and minimum to be.

Example 4.19. Now let's find the maximum and minimum of $f(x, y) = 3x+y$ on $x^2+y^2 = 4$.



We compute that $\nabla f(x, y) = (3, 1)$, so we get $3 = \lambda 2x$ and $1 = \lambda 2y$. This gives us $x = 3y$, and thus we get $10y^2 = 4$, or $y = \pm\sqrt{2/5}$. So our two critical points are $(3\sqrt{2/5}, \sqrt{2/5})$ and $(-3\sqrt{2/5}, -\sqrt{2/5})$.

Plugging in values gives a maximum of $10\sqrt{2/5}$ at $(3\sqrt{2/5}, \sqrt{2/5})$ and a minimum of $-10\sqrt{2/5}$ at $(-3\sqrt{2/5}, -\sqrt{2/5})$.

We sometimes like to express these in terms of the *Lagrangian function*.

Definition 4.20. If we want to optimize $f(x, y)$ subject to $g(x, y) = c$, then the *Lagrangian function* of the problem is

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c).$$

Unconstrained critical points of $\mathcal{L}$ correspond to critical points of the original constrained optimization.

Example 4.21. Suppose we're running a factory, and our output depends on three inputs: our output is $f(x, y, z) = 20x^{3/5}y^{2/5}z^{1/5}$.

Each input has a cost. x costs 50, y costs 30, and z costs 20. If our total budget is \$18,000, how can we maximize the output?

Our budget constraint is $50x + 30y + 20z = 18000$. Then we can set up the Lagrangian:

$$\begin{aligned}
\mathcal{L}(x, y, z, \lambda) &= 20x^{3/5}y^{2/5}z^{1/5} - \lambda(50x + 30y + 20z - 18000)\\
\mathcal{L}_x &= 12\frac{y^{2/5}z^{1/5}}{x^{2/5}} - 50\lambda\\
\mathcal{L}_y &= 8\frac{x^{3/5}z^{1/5}}{y^{3/5}} + 30\lambda\\
\mathcal{L}_z &= 4\frac{x^{3/5}y^{2/5}}{z^{4/5}} + 20\lambda\\
\mathcal{L}_\lambda &= 18000 - 50x - 30y - 20z.
\end{aligned}$$

(We see that we get the constraint equation back as the partial with respect to λ). Solving for λ gives

$$\lambda = \frac{6}{25}\frac{y^{2/5}z^{1/5}}{x^{2/5}}$$
$$\lambda = \frac{4}{15}\frac{x^{3/5}z^{1/5}}{y^{3/5}}$$
$$\lambda = \frac{1}{5}\frac{x^{3/5}y^{2/5}}{z^{4/5}}$$

Solving for $z^{1/5}$ in the first two equations and setting them equal gives

$$\frac{25}{6}\lambda(x/y)^{2/5} = \frac{15}{4}\lambda(y/x)^{3/5}$$
$$50x = 45y$$
$$x = 9y/10.$$

Similarly, we can solve for $x^{3/5}$ and equate the last two equations, which gives

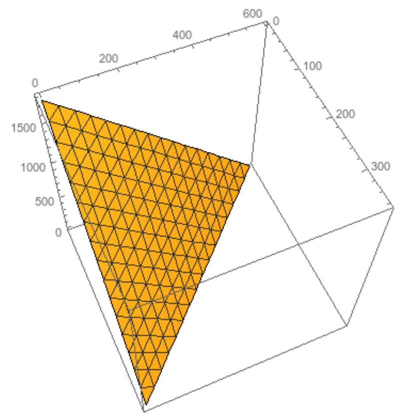
$$\frac{15}{4}\lambda y^{3/5}/z^{1/5} = 5\lambda z^{4/5}/y^{2/5}$$
$$15y = 20z$$
$$z = 3y/4.$$

Plugging this all into the fourth (constraint) equation gives

$$\begin{aligned} 0 &= 18000 - 50(9y/10) - 30y - 20(3y/4) \\ &= 18000 - 45y - 30y - 15y = 20000 - 90y \\ y &= 200. \end{aligned}$$

This also gives us $x = 180$ and $z = 150$.

How do we know this is a maximum? Well, the graph of the constraint is a plane, and the region of possible solutions is a triangle where the plane intersects the $x = 0$, $y = 0$, and $z = 0$ planes (since we can't produce negative amounts). On this entire boundary region the output is zero, and the output at our critical point is $\approx 462 > 0$, so we know that the boundary points are minima and the critical point is a maximum.



This brings us back to the problem of finding global extrema on a region. The basic approach is to look for critical points in the interior, and then use Lagrange multipliers to find any extrema on the boundary.

Example 4.22. Maximize and minimize $f(x, y) = (x-1)^2 + (y-2)^2$ subject to $x^2 + y^2 \leq 45$.

First we look for interior critical points. We have

$$\begin{aligned} f_x(x, y) &= 2(x-1) & x &= 1 \\ f_y(x, y) &= 2(y-2) & y &= 2 \end{aligned}$$

so the unique critical point is at $(1, 2)$.

We could use the second derivative test: we compute

$$\begin{aligned} f_{xx}(x, y) &= 2 & f_{xy} &= 0 \\ f_{yy}(x, y) &= 2 \\ D &= 4 > 0 \end{aligned}$$

and since $D > 0$, $f_{xx} > 0$ we know this is a local minimum.

But we don't actually need to do this since we're just looking for largest and smallest point. So we observe that $f(1, 2) = 0$ and move to the boundary. (It is in fact clear that this is a global minimum, since f is a sum of squares and can never give us a negative output).

On the boundary, we have the constraint $x^2 + y^2 = 45$ and we have $\nabla f(x, y) = (2(x-1), 2(y-2))$ and $\nabla g(x, y) = (2x, 2y)$. So we calculate

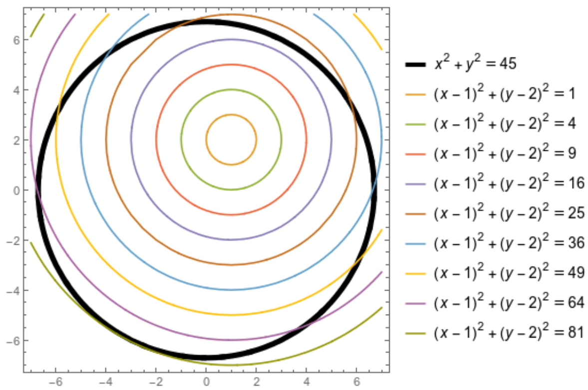
$$\begin{aligned} 2(x-1) &= \lambda 2x & \lambda &= \frac{x-1}{x} \\ 2(y-2) &= \lambda 2y & \lambda &= \frac{y-2}{y} \\ \frac{x-1}{x} &= \frac{y-2}{y} \\ xy - y &= xy - 2x & 2x &= y. \end{aligned}$$

Plugging this into the constraint gives us $5x^2 = 45$ so $x^2 = 9$ and $x = \pm 3$. Then we have $y = \pm 6$. So the two critical points are $(3, 6)$ and $(-3, -6)$.

We calculate

$$f(3,6) = 2^2 + 4^2 = 20 \qquad f(-3,-6) = (-4)^2 + (-8)^2 = 48.$$

Thus the global maximum is 48, achieved at $(-3, -6)$, while the global minimum is 0, achieved at $(1, 2)$.



What does λ mean? It tells us how much the optimum changes when you change the constraint c. Geometrically, we have $\nabla f = \lambda \nabla g$. ∇g is, roughly speaking, how quickly c increases if we move the contour; ∇f is of course how quickly f changes when we move the contour. λ is the ratio between these, and thus how quickly f changes when we move c.

Alternatively, we can compute this with the chain rule. We know that $\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$, and of course $\frac{dg}{dc} = 1$ since $c = g(x, y)$. Then we can compute:

$$\begin{aligned} \frac{df}{dc} &= \frac{\partial f}{\partial x}\frac{dx}{dc} + \frac{\partial f}{\partial y}\frac{dy}{dc} \\ &= \lambda \frac{\partial g}{\partial x}\frac{dx}{dc} + \lambda \frac{\partial g}{\partial y}\frac{dy}{dc} = \lambda \frac{dg}{dc} = \lambda. \end{aligned}$$

Example 4.23. Let's find the global extrema of $f(x, y) = x^2 y + 3y^2 - y$ on $x^2 + y^2 \leq 10$

To find interior critical points, we compute:

$$(2xy, x^2 + 6y - 1) = (0, 0)$$

The first equation tells us that either $x = 0$ or $y = 0$. Thus the critical points are $(0, 1/6), (1, 0)$, and $(-1, 0)$. All three are in the region, so we consider all of them; we get values of $-1/12, 0,$, and 0 respectively.

Now we want to find extrema on the boundary. We compute:

$$\begin{aligned}
f_x &= 2xy = \lambda 2x = g_x \\
f_y &= x^2 + 6y - 1 = \lambda 2y = g_y \\
\lambda &= y \\
x^2 &= 2y^2 - 6y + 1 \\
10 - y^2 &= 2y^2 - 6y + 1 \\
0 &= 3y^2 - 6y - 9 = 3(y^2 - 2y - 3) = 3(y-3)(y+1)
\end{aligned}$$

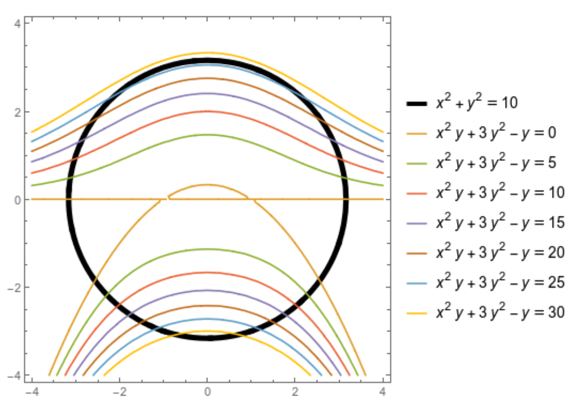
So we get $y = 3$ or $y = -1$. This gives us critical points $(\pm 1, 3)$ and $(\pm 3, -1)$.

$$\begin{aligned}
f(\pm 1, 3) &= 3 + 27 - 3 = 27 \\
f(\pm 3, -1) &= 9 + 3 + 1 = 13
\end{aligned}$$

(Incidentally, we can compute that $\lambda(\pm 1, 3) = 3$, which tells us that at $(1, 3)$, increasing c by 1 would increase the maximum value of f by about 3).

So over the whole region, the global minimum is $-1/12$ at $(0, 1/6)$ and the global maximum is 27 at $(\pm 1, 3)$.

As a note: why do we always get both positive and negative x values for each y? The x variable only shows up in an x^2 so it can never affect anything whether it's positive or negative. We see this represented in the graph, because it is left-right symmetric.



5 Integration

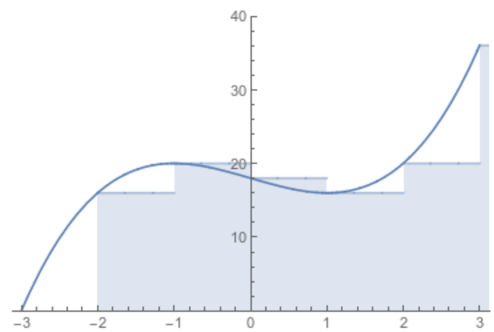
5.1 Riemann sums in multiple variables

Fundamentally, integrals are trying to add up all the value a function has in a given region. We do this by dividing the region up into a bucnh of subregions, estimating the total value in each subregion, and then adding these all back up.

In single-variable calculus we did this with a *Riemann Sum.* You might recall that we defined

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^n f(x_i^*)\Delta x = \lim_{n\to\infty} \sum_{i=1}^n f\left(a + i\frac{b-a}{n}\right)\frac{b-a}{n}.$$

The basic idea here is that we divide the interval $[a, b]$ up into n subintervals. Then we pick some point x_i^* in the subinterval to represent the "average" value in that interval, and estimate the total value to be $f(x_i^*)\Delta x$. We graphically represent this by drawing a rectangle for every subinterval with height $f(x_i^*)$, and adding up the areas of the rectangles.

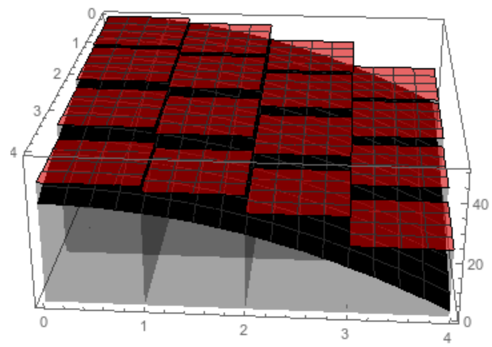


We'd like to do the same thing for a function of two or more variables. We'll stick with a two-variable function for now, and build the same picture. But since our function has two input variables, the geometry becomes three-dimensional. Rather than starting with an interval and dividing it into subintervals, we'll start with a rectangle and divide it into subrectangles.

Definition 5.1. Suppose $f(x, y)$ is continuous on a rectangle $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$. Let (u_{ij}, v_{ij}) be any point in the ijth subrectangle. We define the *definite integral* of f over R to be

$$\int_R f dA = \lim_{\Delta x, \Delta y \to 0} \sum_{i,j} f(u_{ij}, v_{ij})\Delta x \Delta y.$$

If R is a non-rectangular region, we define $\int_R f dA$ similarly, except we ignore any subrectangle not contained in R. We can think of this as treating $f(u_{ij}, v_{ij}) = 0$ if (u_{ij}, v_{ij}) is not in R.



With the single-variable integral, we might worry that it matters *which* choice of value we take, but it turns out that that doesn't matter: in the limit they will converge to the same thing. The same is true in more variables.

Theorem 5.2. *If $f(x, y)$ is continuous and R is bounded, then $\int_R f dA$ converges, and the limit does not depend on the choices of (u_{ij}, v_{ij}).*

Sketch of Proof. If f is continuous on a closed and bounded region, then as Δx and Δy tend to zero, the difference between the maximum and minimum possible values of $f(x, y)$ within each rectangle tend to zero. Thus the largest possible sum and the smallest possible sum will converge to the same point; by the squeeze theorem, any intermediate sum will also converge. □

We can interpret this sum in a couple of different ways. One is volume. In the single-variable case, the integral estimates the (signed) area under the curve. In the multiple variable case, it estimates the volume under the surface given by the graph of the function.

Example 5.3. Suppose we want to estimate the area under the function $f(x, y) = 16 - 3x^2 - y^2$ on the rectangle with corners at $(0, 0)$ and $(2, 2)$. We can divide this up into four subrectangles, each of which is 1×1.

First let's get a definite overestimate, by always taking the highest point in each subrectangle. It's not too hard to see that for f, this will always be the point closest to the origin. So we have

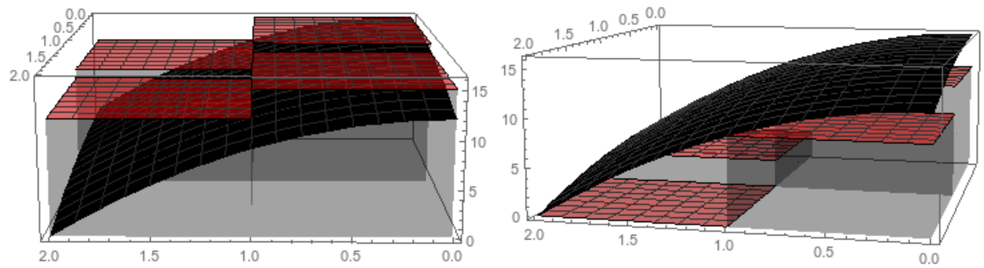
$$\int_R f da \approx f(0,0) \cdot 1 + f(1,0) \cdot 1 + f(0,1) \cdot 1 + f(1,1) \cdot 1 = 16 + 13 + 15 + 12 = 56.$$

We can also get an underestimate by taking the lowest value, which in this case will always be the upper-right point.

$$\int_R f da \approx f(1,1) \cdot 1 + f(2,1) \cdot 1 + f(1,2) \cdot 1 + f(2,2) \cdot 1 = 12 + 3 + 9 + 0 = 24.$$

So we can be pretty sure the volume is somewhere between 24 and 56. We would probably estimate something like $(24 + 56)/2 = 40$.

(If we compute the integral exactly, as we will learn in the next section, we will see that the integral is $\frac{128}{3} \approx 42.67$, so this estimate isn't too bad!)



"Volume under the surface" is a good way to interpret a 2-dimensional integral, but doesn't make much sense of a three-dimensional integral. (We can talk about the "hyper-volume" of the four-dimensional region, but that doesn't give much intuition since we can't really visualize hypervolumes).

Another way of understanding the integral is to think about averages. The integral $\int_R f dA$ is somehow computing the "total" value of f in the region. So we can also compute the *average* value of f in the region to be

$$\text{average} = \frac{1}{\text{Area}(R)} \int_R f dA.$$

This interpretation makes perfect sense in any number of variables we choose.

Definition 5.4. Suppose $f(x, y, z)$ is continuous on a region R, and let $(u_{ijk}, v_{ijk}, w_{ijk})$ be a point in the ijkth sub-prism. Then we define

$$\int_R f dV = \lim_{\Delta x, \Delta y, \Delta z \to 0} \sum_{i,j,k} f(u_{ijk}, v_{ijk}, w_{ijk}) \Delta x \Delta y \Delta z.$$

5.2 Iterated integrals

Computing multivariable integrals by writing out an expression for the Riemann sum and computing the limit is terrible. Fortunately we don't have to do that.

In single-variable calculus, we avoided doing the Riemann sum through the Fundamental Theorem of Calculus, which allowed us to evaluate an antiderivative on the endpoints of an interval, rather than summing the function on the whole interval. That is in fact possible to do here, but is somewhat complex, since the boundary of a two-dimensional region has infinitely many points. We'll return to this idea towards the end of class. But for right now, we'll do something much simpler.

When we wrote down the definition of a two-variable Riemann sum, we just said to add up the values for all the subrectangles; we didn't say anything about what order to add them up in. And as long as the sum is finite, this can't possibly matter.

For infinite sums, the order you add things up in can matter (see e.g. the Riemann Series Theorem if you want to know more about this). But fortunately, it turns out that in this case it does not.

Theorem 5.5 (Fubini)**.** *Let $R = \{(x,y) : a \leq x \leq b, c \leq y \leq d\} = [a,b] \times [c,d]$, and let $f(x,y)$ be continuous on R. Then*

$$\int_R f dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

This means that rather than somehow doing the "whole" double integral, we can do two single-variable integrals in succession. And we already know how to do those!

Example 5.6. Let $R = \{(x,y) : 1 \leq x \leq 4, 0 \leq y \leq 3\}$ and let $f(x,y) = xy^2$. Then we can compute

$$\begin{aligned}\int_R f\,dA &= \int_1^4 \int_0^3 xy^2\,dy\,dx = \int_1^4 \left(xy^3/3\big|_0^3\right)\,dx \\ &= \int_1^4 9x\,dx = 9x^2/2\big|_1^4 = 72 - 9/2 = 135/2.\end{aligned}$$

Alternatively, we could compute:

$$\begin{aligned}\int_R f\,dA &= \int_0^3 \int_1^4 xy^2\,dx\,dy = \int_0^3 \left(x^2/2y^2\big|_1^4\right)\,dy \\ &= \int_0^3 8y^2 - y^2/2\,dy = \int_0^3 15y^2/2\,dy = 15y^3/6\big|_0^3 = 135/2.\end{aligned}$$

Notice we get the same answer with either order of integration.

Example 5.7. Suppose we have a building with a corrugated sine-wave roof. It is 6 meters wide and 8 meters long. The corners are 2 and 3 meters high, and along the length the sine wave oscillates four times. What is the volume of the building?

The height is given by $f(x, y) = 2 + x/6 + \sin(\pi y)$. Then the volume is given by

$$\begin{aligned}
\int_0^6 \int_0^8 2 + x/6 + \sin(\pi y)\, dy\, dx &= \int_0^6 \left(2y + xy/6 - \cos(\pi y)/\pi\big|_0^8\right)\, dx \\
&= \int_0^6 16 + 4x/3 - 1/\pi - (0 + 0 - 1/\pi)\, dx \\
&= \int_0^6 16 + 4x/3\, dx = 16x + 2x^2/3|_0^6 = 96 + 24 = 120.
\end{aligned}$$

Integrals of three-variable functions work exactly the same way that integrals of two variables work. We just have three iterated integrals instead of two.

Example 5.8. Suppose we have a box that has a 3 inch square base, and is 4 inches tall, and has a density of $1 + xy + yz + xz^2$ ounces per cubic inch. What is the total mass?

We want to compute the integral of $f(x, y, z) = 1 + xy + yz + xz^2$ over this rectangular box. So we compute

$$\begin{aligned}
M &= \int_0^3 \int_0^3 \int_0^4 1 + xy + yz + xz^2\, dz\, dy\, dx = \int_0^3 \int_0^3 z + xyz + yz^2/2 + xz^3/3|_0^4\, dy\, dx \\
&= \int_0^3 \int_0^3 4 + 4xy + 8y + 64x/3\, dy\, dx = \int_0^3 4y + 2xy^2 + 4y^2 + 64xy/3|_0^3\, dx \\
&= \int_0^3 12 + 18x + 36 + 64x\, dx = 48x + 41x^2|_0^3 = 513.
\end{aligned}$$

Thus the box has a mass of 513 ounces.

We can also use iterated integrals to integrate over non-rectangular (or non-box) regions. In this case we'll let x (say) vary from its minimum possible value to its maximum possible value; but for each x, the possible y values will depend on the current x value.

Example 5.9. Integrate the function $f(x, y) = xy$ over the triangle with corners at $(0, 0)$, $(1, 0)$, and $(1, 3)$.

We have x varying from 0 to 1. The upper bound of the triangle is given by the line $y = 3x$, so the y bounds are from 0 to $3x$. Thus we have the double integral

$$\begin{aligned}
\int_0^1 \int_0^{3x} xy\, dy\, dx &= \int_0^1 xy^2/2|_0^{3x}\, dx = \int_0^1 9x^3/2\, dx \\
&= 9x^4/8|_0^1 = 9/8.
\end{aligned}$$

We could just as easily have done it the other way. y varies from 0 to 3, and x varies from $y/3$ to 1. So we have the double integral

$$\begin{aligned}
\int_0^3 \int_{y/3}^1 xy\, dx\, dy &= \int_0^3 x^2y/2|_{y/3}^1\, dy = \int_0^3 (y/2 - y^3/18)\, dy \\
&= y^2/4 - y^4/72|_0^3 = 9/4 - 9/8 = 9/8.
\end{aligned}$$

Thus we get the same answer integrating either way.

Example 5.10. Let's integrate the function $f(x, y) = y\sqrt{x}$ over the parallelogram with corners at $(0, 1), (0, 2), (1, 0), (1, 1)$.

We see that x varies from 0 to 1, and y varies from $1 - x$ to $2 - x$. So we have

$$\begin{aligned}\int_0^1 \int_{1-x}^{2-x} y\sqrt{x}\,dy\,dx &= \int_0^1 2y^2\sqrt{x}/2|_{1-x}^{2-x}\,dx = \int_0^1 (2-x)^2\sqrt{x}/2 - (1-x)^2\sqrt{x}/2\,dx \\ &= \int_0^1 (4 - 4x + x^2 - 1 + 2x - x^2)\sqrt{x}/2\,dx = \int_0^1 3/2\sqrt{x} - x^{3/2}\,dx \\ &= x^{3/2} - 2/5x^{5/2}|_0^1 = 3/5.\end{aligned}$$

Could we integrate the other way? Sure. But it's actually a big pain, since writing x as a function of y would have to go piecewise: we'd get something like

$$x = \begin{cases} 1 - y \leq x \leq 1 & y \leq 1 \\ 0 \leq x \leq 2 - y & 1 \leq y \leq 2 \end{cases}$$

So we'd have to set up and evaluate two separate integrals here, and get something like

$$\int_0^1 \int_{1-y}^1 y\sqrt{x}\,dx\,dy + \int_1^2 \int_0^{2-y} y\sqrt{x}\,dy\,dx.$$

Integrating by y and then x is very much the correct choice here.

Remark 5.11. Whenever setting up an iterated integral, remember that the final answer should be a number. Therefore the bounds of the outer integral should always be constants. The bounds on the inner integrals can depend on variables from integrals to the outside, but not on variables from integrals to the inside.

At each step, you should have one fewer variable to worry about (although possibly a more complex algebraic expression).

Example 5.12. Find the volume of the region bounded by $z = x+y, z = 10$, and the planes $x = 0, y = 0$.

We can set this up as a two-variable integral or as a three-variable integral. As a two-variable integral we'd need the region in the plane and the height. The solid exists over a region bounded by $0 \leq x \leq 10$ and $0 \leq y \leq 10-x$. Then the height is given by the difference between $z = 10$ and $z = x + y$, so we have $f(x, y) = 10 - x - y$. Then we get the integral

$$\begin{aligned}
\int_0^{10} \int_0^{10-x} 10 - x - y \, dy \, dx &= \int_0^{10} 10y - xy - y^2/2|_0^{10-x} \, x \\
&= \int_0^{10} 100 - 10x - 10x + x^2 - (100 - 20x + x^2)/2 \, dx \\
&= \int_0^{10} 50 - 10x + x^2/2 \, dx = 50x - 5x^2 + x^3/6|_0^{10} \\
&= 500 - 500 + 500/6 = 500/6.
\end{aligned}$$

But it's actually a bit more natural to express this as a triple integral. The volume of a region is just the integral of the function 1 over that region. So we can write

$$\begin{aligned}
V &= \int_0^{10} \int_0^{10-x} \int_{x+y}^{10} dz \, dy \, dx \\
&= \int_0^{10} \int_0^{10-x} z|_{x+y}^{10} \, dy \, dz \\
&= \int_0^{10} \int_0^{10-x} 10 - x - y \, dy \, dx.
\end{aligned}$$

This of course gets us the same answer as before, but is often a bit easier to think about.

Example 5.13. Let's find the average value of the function $f(x, y) = y(x-1)$ on the region bounded by $y = x$ and $y = x^2$.

We can set up $0 \leq x \leq 1$ and $x^2 \leq y \leq x$. So we get the integral

$$\begin{aligned}
\int_0^1 \int_{x^2}^{x} y(x-1) \, dy \, dx &= \int_0^1 y^2/2(x-1)|_{x^2}^{x} \, dx = \int_0^1 x^3/2 - x^2/2 - (x^5/2 - x^4/2) \, dx \\
&= \int_0^1 -x^5/2 + x^4/2 + x^3/2 - x^2/2 \, dx = -x^6/12 + x^5/10 + x^4/8 - x^3/6|_0^1 \\
&= -1/12 + 1/10 + 1/8 - 1/6 = -1/40.
\end{aligned}$$

That gives us the total value of the function. We could also do the integral in the opposite order: we have $0 \leq y \leq 1$ and $y \leq x \leq \sqrt{y}$, and we get

$$\begin{aligned}\int_0^1 \int_y^{\sqrt{y}} y(x-1)\,dx\,dy = \int_0^1 x^2y/2 - xy|_y^{\sqrt{y}}\,dy = \int_0^1 y^2/2 - y^{3/2} - (y^3/2 - y^2)\,dy \\ = \int_0^1 -y^3/2 + 3y^2/2 - y^{3/2}\,dy = -y^4/8 + y^3/2 - 2/5y^{5/2}|_0^1 \\ = -1/8 + 1/2 - 2/5 = -1/40.\end{aligned}$$

But we wanted the average, so we still need the area of the region. This is basically a single-variable calculus question: we integrate the height of the region over the interval. But we can also set it up as a multivariable integral: we just integrate the function "1" over the region. We get

$$\begin{aligned}A = \int_0^1 \int_{x^2}^x 1\,dy\,dx = \int_0^1 y|_{x^2}^x\,dx = \int_0^1 x - x^2\,dx \\ = x^2/2 - x^3/3|_0^1 = 1/2 - 1/3 = 1/6.\end{aligned}$$

(It's not an accident that the integral at the end of line 1 is exactly what you'd get by setting this up as a single-variable integral in Calculus 2).

Thus the total value is $-1/40$, and the area is $1/6$. So the average value of f on the region is

$$\text{average} = \frac{1}{\text{area}} \int_R f\,dA = 6(-1/40) = -3/20.$$

Example 5.14. Set up an integral to find the mass of a solid cone bounded by the xy plane and the cone $z = 4 - \sqrt{x^2+y^2}$, if the density is given by $\delta(x,y,z) = xz$.

We have the iterated integral

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^3}} \int_0^{4-\sqrt{x^2+y^2}} xz\,dz\,dy\,dx.$$

Example 5.15. Set up an integral to find the volume o the solid below the graph of $f(x,y) = 25 - x^2 - y^2$ and above the plane $z = 9$.

The two surfaces intersect where $x^2 + y^2 = 16$. We can either write the double integral

$$\int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 16 - x^2 - y^2\,dy\,dx$$

or we can write the triple integral

$$\int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_9^{25-x^2-y^2} dz\,dy\,dx$$

Example 5.16. Set up an integral to find the volume of the region in the first octant bounded by the coordinate planes, the plane $z = 3$, and the surface $z = x^2 + y^2$.

We can see we have z varying from 0 to 3. For each z, we have x varying from 0 to $\sqrt{z}$, and then y varying from 0 to $\sqrt{z - x^2}$. So we get the integral

$$\int_0^3 \int_0^{\sqrt{z}} \int_0^{\sqrt{z-x^2}} 1 \, dy \, dx \, dz.$$

In most of these cases we have a few different options for how to set up the integral. So far these choices haven't mattered that much, but sometimes they matter a great deal.

Example 5.17.

$$\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1} \, dy \, dx.$$

The integral with respect to y is a huge pain, so we don't do it. We sketch the region: x goes from 0 to 6, and y goes from $x/3$ to 2. We can turn this around to say: y goes from 0 to 2, and x goes from 0 to $3y$. So we get

$$\begin{aligned}
\int_0^2 \int_0^{3y} x\sqrt{y^3+1} \, dx \, dy &= \int_0^2 \left(x^2/2\sqrt{y^3+1}\Big|_0^{3y} \right) \, dy \\
&= \int_0^2 \left(9y^2/2\sqrt{y^3+1} \right) \, dy \\
&= (y^3+1)^{3/2}|_0^2 = 27 - 1 = 26.
\end{aligned}$$

Example 5.18.

$$\begin{aligned}
\int_0^2 \int_y^2 e^{x^2} \, dx \, dy &= \int_0^2 \int_0^x e^{x^2} \, dy \, dx \\
&= \int_0^2 xe^{x^2} \, dx = e^{x^2}/2|_0^2 = e^4/2 - 1/2.
\end{aligned}$$

And sometimes, no matter what you do, the integral will be gross.

Example 5.19. Find the mass of the solid bounded by the xy plane, the yz plane, the xz plane, and the plane $x + 3y + 2z = 6$, if the density is given by $\delta(x, y, z) = x + z$.

We see that x varies from 0 to 6, and then z varies from 0 to $(6-x)/2$, and then y varies from 0 to $(6 - x - 2z)/3$. So we get

$$M = \int_0^6 \int_0^{3-x/2} \int_0^{2-x/3-2z/3} x + z \, dy \, dz \, dx = 27/2.$$

Example 5.20. Integrate $f(x,y) = x^2y$ over the upper half of the unit circle.

We have that $-1 \leq x \leq 1$ and $0 \leq y \leq \sqrt{1-x^2}$. So we get

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} x^2y\,dy\,dx &= \int_{-1}^{1} x^2y^2/2|_0^{\sqrt{1-x^2}}\,dx \\ &= \int_{-1}^{1} x^2(1-x^2)/2\,dx = \int_{-1}^{1} x^2/2 - x^4/2\,dx \\ &= x^3/6 - x^5/10|_{-1}^{1} = 1/6 - 1/10 + 1/6 - 1/10 = 2/15.\end{aligned}$$

Example 5.21. Integrate $f(x,y) = x^2y^2$ over the upper half of the unit circle. We have that $-1 \leq x \leq 1$ and $0 \leq y \leq \sqrt{1-x^2}$. So we get

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} x^2y^2\,dy\,dx &= \int_{-1}^{1} x^2y^3/3|_0^{\sqrt{1-x^2}}\,dx \\ &= \int_{-1}^{1} x^2(1-x^2)^{3/2}/3\,dx\end{aligned}$$

and this has suddenly become a huge mess—much worse than the previous problem. We can use trigonometric substitution plus some grindy arguments to find that this is equal to

$$\frac{1}{144}\left(x\sqrt{1-x^2}(-8x^4+14x^2-3)+3\arcsin(x)\right)\Big|_{-1}^{1} = \frac{\pi}{48},$$

but ultimately there's nothing we can do to this integral that will make it nice.

The fundamental problem in this last exampleis that since we're integrating over a circle, we have these $\sqrt{1-x^2}$ terms that we just can't get rid of.

Unless we develop a completely different approach to setting up integrals, that somehow is more compatible with circles.

5.3 Integrals in Polar Coordinates

Describing circles in Cartesian coordinates is fundamentally a bit awkward. It's much easier to describe a circle or circle-like region in terms of polar coordinates.

Definition 5.22. The *polar coordinates* of a point $P \in \mathbb{R}^2$ are a pair of numbers (r,θ), where r is the distance between P and the origin O, and θ is the angle between the vector $\vec{i}$ and the vector $\overrightarrow{OP}$.

We always choose these numbers so that r is positive, and $\theta \in [0, 2\pi)$.

Proposition 5.23. *Suppose (x,y) are the cartesian coordinates of a point P, and $(r\,theta)$ are the polar coordinates. Then:*

- $x = r\cos\theta$
- $y = r\sin\theta$
- $r = \sqrt{x^2+y^2}$
- $\theta = \pm\arctan y/x$.

Example 5.24. The polar equation for a circle of radius c is $r = c$. The closed disk of radius c is given by the set $\{(r,c) : 0 \leq r \leq c, 0 \leq \theta < 2\pi\}$. The Cartesian coordinates are $\{(x,y) : x^2+y^2 \leq c^2\}$.

The wedge of the closed unit disk in the first (upper-right) quadrant is $\{(r,\theta) : 0 \leq r \leq 1, 0 \leq \theta \leq \pi/2\}$. The Cartesian coordinates are $\{(x,y) : x \geq 0, y \geq 0, x^2+y^2 \leq 1.\}$

The set $\{(r,\theta) : 1 \leq r \leq 2, \pi \leq \theta \leq 3\pi/2\}$ is a wedge of an annulus with inner radius 1 and outer radius 2, in the third (lower-left) quadrant. The Cartesian coordinates here are $\{(x,y) : x \leq 0, y \leq 0, 1 \leq x^2+y^2 \leq 4\}$.

The polar equation for the line $y = 2x$ is $r\sin\theta = 2r\cos\theta$, which reduces to $\sin\theta = 2\cos\theta$.

Notice that all the circle equations become much simpler than their cartesian equivalents, but the line (and anything else rigid and rectangular) becomes much more complex.

We want to exploit this complexity reduction to make integrals of functions over circular regions easier. When we integrated over a rectangular region, we did this by dividing the region into rectangles. Using polar coordinates to integrate over a circular or wedge-like region, we'll divide the region into *subwedges*.

What is the area of a wedge? Each wedge is *roughly* a rectangle. (This is very rough, but in the limit it all washes out). The thickness of the rectangle is the change in the radius, so we call that dr. The width of the rectangle is *proportional* to the change in angle, but not equal to it: by definition, an arc of θ radians has a length of θr. Thus the width of our rectangle is $r\,d\theta$, the change in the angle times the actual radius.

This if we want to integrate a function in polar coordinates, we use the formula

$$I = \int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.$$

Note the extra r in the formula! This is very important, and converts a number of integrals from "obnoxious" to "easy".

Example 5.25. Let's integrate $f(x,y) = x^2y$ over the upper half of the unit circle.

We see that this is a region given by $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi$. So we compute

$$\begin{aligned} I &= \int_0^1 \int_0^\pi r^2 \cos^2 \theta r \sin \theta r \, d\theta \, dr \\ &= \int_0^1 \int_0^\pi r^4 \cos^2 \theta \sin \theta \, d\theta \, dr \\ &= \int_0^1 r^4 \frac{-1}{3} \cos^3 \theta |_0^\pi \, dr = \int_0^1 r^4 \frac{2}{3} \, dr \\ &= \frac{2}{15} r^5 |_0^1 = \frac{2}{15}. \end{aligned}$$

Example 5.26. What about $f(x, y) = x^2 y^2$ over that same region? We have

$$\begin{aligned} I &= \int_0^1 \int_0^\pi r^2 \cos^2 \theta r^2 \sin^2 \theta r \, d\theta \, dr \\ &= \int_0^1 \int_0^\pi r^5 \cos^2 \theta \sin^2 \theta \, d\theta \, dr \\ &= \int_0^1 r^5 \left(\frac{\theta}{8} - \frac{1}{32} \sin(4\theta) \right) \Big|_0^\pi \, dr \\ &= \int_0^1 \frac{\pi r^5}{8} \, dr = \left. \frac{\pi r^6}{48} \right|_0^1 = \frac{\pi}{48}. \end{aligned}$$

And while I didn't actually show the work to do that first antiderivative, it's a standard calc 2 trick—unlike the non-polar version, which is basically undoable.

Some functions also become much easier to integrate in polar coordinates.

Example 5.27. Integrate the function $f(x, y) = (x^2 + y^2)^{-1/2}$ over the annulus with inner radius 1 and outer radius 2.

We have bounds $1 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$. More importantly, we see that $f(x, y) = (x^2 + y^2)^{-1/2} = (r^2)^{-1/2} = \frac{1}{r}$. Thus we have

$$\begin{aligned} I &= \int_0^{2\pi} \int_1^2 \frac{1}{r} \, dr \, d\theta = \int_0^{2\pi} \ln |r| |_1^2 \, d\theta \\ &= \int_0^{2\pi} \ln 2 \, d\theta = 2\pi \ln 2. \end{aligned}$$

Example 5.28. Let's find the area of the spiral that has thickness 1, and has inner radius going from 0 to 1 over one complete rotation.

$$\begin{aligned}
\int_0^{2\pi} \int_{\theta/(2\pi)}^{1+\theta/(2\pi)} r\, dr\, d\theta &= \int_0^{2\pi} \frac{r^2}{2}\bigg|_{\theta/(2\pi)}^{1+\theta/(2\pi)} d\theta \\
&= \frac{1}{2}\int_0^{2\pi} (1+\theta/(2\pi))^2 - (\theta/(2\pi))^2\, d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 1 + \theta/\pi + \theta^2/(4\pi^2) - \theta^2/(4\pi^2)\, d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 1 + \theta/pi\, d\theta = \frac{1}{2}\left(\theta + \theta^2/(2\pi)|_0^{2\pi}\right) \\
&= \frac{1}{2}\left(2\pi + (2\pi)^2/(2\pi)\right) = 2\pi.
\end{aligned}$$

5.4 Cylindrical and Spherical Coordinates

We can extend this idea to three dimensions. There are two different ways to do this, which are suited to different types of regions.

Definition 5.29. The *cylindrical coordinates* of a point $P \in \mathbb{R}^3$ are a triple of numbers (r, θ, z), where r is the distance between the origin O and the projection of P into the xy plane; and θ is the angle between the vector $\vec{i}$ and the projection of $\overrightarrow{OP}$ into the xy plane; and z is the height.

We always choose these numbers so that r is positive, and $\theta \in [0, 2\pi)$.

Proposition 5.30. *Suppose (x, y, z) are the cartesian coordinates of a point P, and $(r\, theta, h)$ are the polar coordinates. Then:*

- $x = r\cos\theta$
- $y = r\sin\theta$
- $r = \sqrt{x^2 + y^2}$
- $\theta = \pm \arctan y/x$
- $z = h.$

We can work out the integral formula here, just like we did for polar integrals. We divide our region into three-dimensional wedges—imagine a wedge of cheese. Each wedge is *roughly* a rectangular prism, as in polar integrals. The area of the base of the wedge is still $r\, dr\, d\theta$, and the height is dz, so when we do our integrals in cylindrical coordinates, we integrate $f(r, \theta, z) r \, dr \, d\theta \, dz$.

Example 5.31. Integrate xz over wedge cut from cylinder 4 cm high and 6 cm in radius, angle $\pi/6$ above x axis.

$$\int_0^4 \int_0^6 \int_0^{\pi/6} r \cos\theta z r \, d\theta \, dr \, dz = 288$$

Example 5.32. Integrate the function xyz over the cone bounded by $0 \leq z \leq 4$ and $x^2 + y^2 = z^2$ and the plane $z = 0$.

$$\int_0^4 \int_0^z \int_0^{2\pi} r^3 \cos\theta \sin\theta z \, d\theta \, dr \, dz = 0$$

Example 5.33. Set up an integral in spherical coordinates to find the volume inside the unit sphere.

$$\int_0^{2\pi} \int_{-1}^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} dz \, dr \, d\theta.$$

As we can see, cylindrical coordinates are still pretty unsuited to describing actual spheres. For those, we want to use a different coordinate system entirely.

Definition 5.34. The *spherical coordinates* of a point $P \in \mathbb{R}^3$ are a triple of numbers (ρ, θ, ϕ), where ρ is the distance between the origin O and the point P; θ is the angle between the vector $\vec{i}$ and the projection of $\overrightarrow{OP}$ into the xy plane; and ϕ is the angle between the vector $\overrightarrow{OP}$ and the vector $\vec{k}$.

We always choose these numbers so that ρ is positive, $\theta \in [0, 2\pi)$, and $\phi \in [0, \pi]$.

Proposition 5.35. *Suppose (x, y, z) are the cartesian coordinates of a point P, and $(r\, theta, h)$ are the polar coordinates. Then:*

- $x = \rho \sin(\phi) \cos(\theta)$
- $y = \rho \sin(\phi) \sin(\theta)$
- $z = \rho \cos(\phi)$
- $\rho^2 = x^2 + y^2 + z^2$.

Next we need the integral formula. We again divide our region into wedges, but these are wedges of a spherical shell, rather than the blocks-of-cheese that feature in cylindrical coordinates.

Again the thickness is just $d\rho$. We need to compute the area of the inner square of the wedge. We see that the "height" is determined by the length of the ϕ arc, and thus is $\rho\, d\phi$.

The "width" is given by the length of the θ arc. In cylindrical coordinates this was given by $r\, d\theta$, but we'll have something smaller in spherical coordinates: as you move away from the $z = 0$ plane (which is also the $\phi = \pi/2$ plane!) the radius of the circle given by intersecting the plane $z = z_0$ with the sphere $\rho = \rho_0$ will decrease, proportionately to $\sin\phi$. Thus the height of the wedge is $\rho \sin\phi\, d\theta$, and our integral is

$$\int_{\rho_1}^{\rho_2} \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} f\left(\rho \sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)\right) \rho^2 \sin\phi\, d\phi\, d\theta\, d\rho.$$

Example 5.36. Let's find the volume of the unit sphere. We have

$$\begin{aligned}
\int_0^1 \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin(\phi)\, d\phi\, d\theta\, d\rho &= \int_0^1 \int_0^{2\pi} -\rho^2 \cos(\phi)|_0^{\pi}\, d\theta\, d\rho \\
&= \int_0^1 \int_0^{2\pi} 2\rho^2\, d\theta\, d\rho \\
&= \int_0^1 4\pi\rho^2\, d\rho \\
&= \frac{4}{3}\pi\rho^3|_0^1 = 4\pi/3.
\end{aligned}$$

Example 5.37. Find the mass of a sphere with radius 3 and density equal to $\rho\cos^2\theta$.

$$\begin{aligned}
\int_0^3 \int_0^{2\pi} \int_0^{\pi} \rho^3 \cos^2\theta \sin\phi\, d\phi\, d\theta\, d\rho &= \int_0^3 \int_0^{2\pi} \rho^3 \cos^2\theta(-\cos\phi)|0^{\pi}\, d\theta\, d\rho \\
&= \int_0^3 \int_0^{2\pi} 2\rho^3 \cos^2\theta\, d\theta\, d\rho \\
&= \int_0^3 \rho^3 \left(\theta + \frac{\sin(2\theta)}{2}\, |)\right|_0^{2\pi} d\rho \\
&= \int_0^3 2\pi\rho^3\, d\rho = \pi\rho^4/2|_0^3 = \frac{81\pi}{2}.
\end{aligned}$$

6 Parametrization and Vector Fields

So far in this course we've discussed two types of functions. Single-variable functions are the ones you're familiar with from single-variable calculus; multi-variable functions, like the ones we've been studying, take in multiple variables, but output one.

However, we can also have functions that output more than one variables. In this section we will lay out various types of functions that output multiple variables, and talk about what situationst hey are used to describe. In future sections we will apply calculus ideas to them.

6.1 Curves and Motion

In this section we want to study curves through space. By a curve we mean, essentially, any shape that is in some sense "one-dimensional". So a line, a circle, and a curving spiral through three-dimensional space are all curves.

The essence of a curve is the one-dimensionality. We capture this idea by requiring position on our curves to be described by one single real number. That is, we can describe our position on the curve with exactly one coordinate. We say a system of coordinates for an object is a "parametrization", because it describes the object with some number of parameters.

Definition 6.1. We say a function $\vec{r} : \mathbb{R} \to \mathbb{R}^n$ is a *parametrization of a curve*.

Sometimes we want to consider the *components* of the function. We will usually write $\vec{r}(t) = (x(t), y(t), z(t))$ and say that the single-variable functions $x(t), y(t), z(t)$ are the components of $\vec{r}$.

Example 6.2. Let's find a parametrization for the curve $y = x^2$.

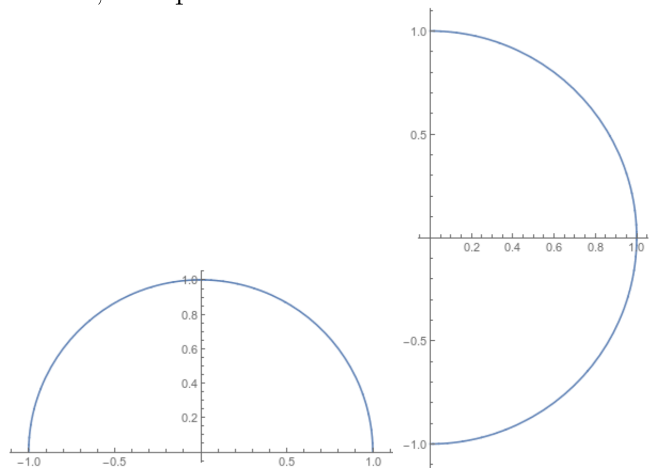
We see that we can parametrize this by the function $\vec{r}(t) = (t, t^2)$. You'll notice that this is basically the original function formula: we have $x = t$ and $y = t^2 = x^2$. Any time we have a curve that is the graph of a function, we effectively have a parametrization for free; the input variable gives us a parametrization.

Example 6.3. Let's parametrize a circle of radius 1. Notice that we can't use the same trick as last time, since this isn't a function.

We could try something like $x(t) = t, y(t) = \sqrt{1 - t^2}$ for $-1 \leq t \leq 1$. This sort of works, but only captures the top half of the circle. We could keep trying to make this idea work, but it baically won't.

Instead, we take advantage of the fact that circles are fundamentally trigonometric. We see that $\vec{r}(t) = (\cos(t), \sin(t))$ will give us every point on the circle—in fact, this is the usual unit circle definition of sin and cos. In particular, we have $\vec{r}(0) = (1, 0)$ is the rightmost point of the circle, and as t increases we move counterclockwise around the circle.

However, this isn't the only possible parametrization. For instance, we could instead take $\vec{s}(t) = (\sin(t), \cos(t))$. This will still parametrize the circle, but it starts at $\vec{s}(0) = (0, 1)$ which is the top of the circle, and proceeds clockwise.

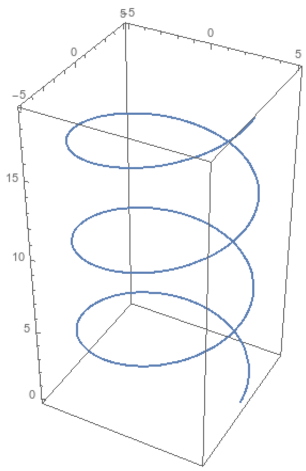


The graphs of $\vec{r}$ and $\vec{s}$ for $0 \leq t \leq \pi$

In general, choices of parametrization aren't unique. Often we can make a problem easier (or harder) by changing our cohice of coordinates.

Example 6.4. Let's consider the curve given by $\vec{r}(t) = (5\cos t, 5\sin t, t)$. This gives us a circle of radius 5 if we consider only the x and y coordinates, but now the z coordinate is increasing. Thus we are spiraling around as our z coordinate increases. This gives us a shape caled a "helix".

(You might recall that DNA is described as a "double helix". This is because it is two of these helixes spiraling around each other).



The two most common shapes to parametrize are probably circles and lines. We've looked at circles already; now let's consider lines.

Example 6.5. Let's parametrize the line through $(1,3,5)$ in the direction of $2\vec{i} - \vec{j} + 3\vec{k}$.

This is simple and straightforward. We get $\vec{r}(t) = (1,3,5) + t(2,-1,3) = (1+2t, 3-t, 5+3t)$

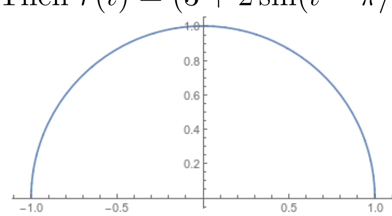
In general, a line is described by a point and a direction. Therefore, if we want to parametrize a line, we can use the equation $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ where $\vec{r}_0$ is the known point and $\vec{v}$ is the direction.

Example 6.6. Another nice property of parametrizations is that it's easy to shift them in space. Let's parametrize a circle of radius 2 centered at $(3,2)$, going counterclockwise starting from the right-hand point.

We know that a circle of radius 1 centered at the origin is $\vec{r}(t) = (\cos(t), \sin(t))$. To get radius 3, we multiply by 3; then to shift the center, we add $(3,2)$, leaving us with the parametrization $\vec{r}(t) = (3 + 2\cos(t), 2 + 2\sin(t))$.

If we want to start from left-hand point and go clockwise, we can do a couple things. One is to flip the circle upside down and start halfway around; this would give $\vec{r}(t) = (3 + 2\cos(t+\pi), 2 - 2\sin(t+\pi))$.

Alternatively, we could start from the parametrization $(\sin(t), \cos(t))$, which already goes clockwise. Then we would get Then $\vec{r}(t) = (3 + 2\sin(t - \pi/2), 2 + 2\cos(t - \pi/2))$.

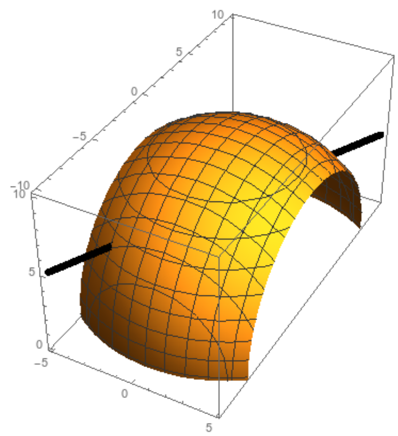


We can use parametrizations of curves to find where they intersect surfaces.

Example 6.7. Where does the curve $(t, 2t, t+3)$ intersect the sphere of radius 9 centered at the origin?

$$\begin{aligned} x^2 + y^2 + z^2 &= 81 \\ t^2 + 4t^2 + t^2 + 6t + 9 &= 81 \\ 6t^2 + 6t &= 72 \\ t^2 + t - 12 &= 0 \\ (t+4)(t-3) &= 0 \end{aligned}$$

and we have $t = 3$ and $t = -4$. Thus our line intersects the sphere at $(3, 6, 6)$ and $(-4, -8, -1)$.

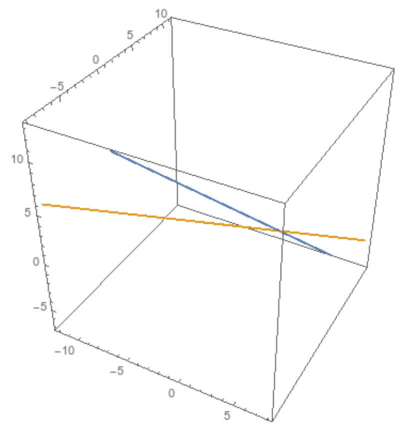


We can also see when (and if) two curves intersect.

Example 6.8. We have particles moving along the paths $\vec{r}_1(t) = (t, 1+2t, 3-2t)$ and $\vec{r}_2 = (-2-2t, 1-2t, 1+t)$. Do the particles hit each other? Do their paths cross?

For the particles to hit each other, we need them to have the same coordinates at the same time. We see they share x coordinates when $t = -2/3$; they share y coordinates when $t = 0$ and they share z coordinates when $t = 2/3$. Thus they never collide.

If we want to see if their paths cross, we just need to test whether they ever pass through the same point. So we solve the system $t_1 = -2 - 2t_2, 1 + 2t_1 = 1 - 2t_2, 3 - 2t_1 = 1 + t_2$. The second equation gives us $t_2 = -t_1$. The other equations then are $t_1 = 2t_1 - 2$ and $3 - 2t_1 = 1 - t_1$; we can see that these both give us $t_1 = 2$ (and thus $t_2 = -2$). So the paths do cross.

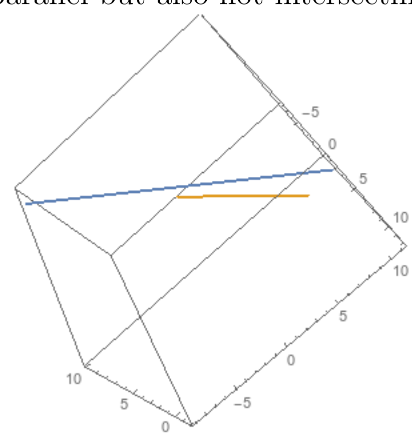


Remark 6.9. Here we have three equations in two variables, so it's very easy for the paths not to cross. But it's also quite possible for them to cross in two or more points.

Finally, we can test the relationship lines have to each other.

Example 6.10. $\vec{r}_1 = (t-1, 1+2t, 5-t)$ and $\vec{r}_2 = (2+2t, 4+t, 3+t)$.

Not parallel, because vectors/slopes are $(1, 2, -1)$ and $(2, 1, 1)$. Also don't intersect, because no solutions. So not parallel but also not intersecting. ("skew").



So far we've discussed parametric equations as giving position as a function of time, and talking about the direction and sometimes the speed of motion. As in the single-variable case, we can make this more precise by the theory of derivatives.

Speed is change in position with respect to time. We can define this pretty easily:

Definition 6.11. The *velocity* of an object that moves along a path with position $\vec{r}(t)$ at time t is

$$\vec{v}(t) = \vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

This definition by itself is a bit hard to work with. However, we can make it much simpler by realizing that the x, y, and z coordinates all change independently, so we can consider them independently. (This is implicitly because derivatives are always linear, so we can write the derivative of a sum as the sum of the derivatives).

Proposition 6.12. *Let $\vec{r} : \mathbb{R}^2 \to \mathbb{R}^3$ be a differentiable function. Then*

$$\vec{r}\,'(t) = (x'(t), y'(t), z'(t)).$$

Proof.

$$\begin{aligned}
\vec{r}\,'(t) &= \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \\
&= \lim_{h\to 0} \frac{x(t+h)\vec{i} + y(t+h)\vec{j} + z(t+h)\vec{k} - x(t)\vec{i} - y(t)\vec{j} - z(t)\vec{k}}{h} \\
&= \lim_{h\to 0} \frac{x(t+h) - x(t)}{h}\vec{i}\frac{y(t+h) - y(t)}{h}\vec{j} + \frac{z(t+h) - z(t)}{h}\vec{k} \\
&= x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}.
\end{aligned}$$

□

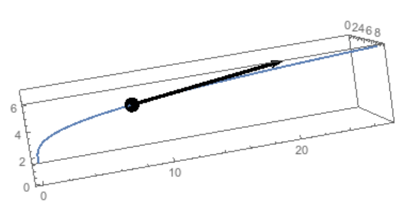
Example 6.13. Consider the circle parametrized by $(\cos(t), \sin(t))$. Then the derivative is $\vec{r}\,'(t) = (-\sin(t), \cos(t))$.

If we want to find the tangent vector at the point $(1, 0)$, we compute the derivative and plug in $t = 0$, so we get $\vec{r}\,'(0) = (0, 1)$ as your vector, and the tangent line is $(1, 0 + t)$.

Now suppose want the tangent line at $(\sqrt{2}/2, \sqrt{2}/2)$. This occurs at time $t = \pi/4$ and so we compute $\vec{r}\,'(\pi/4) = (-\sqrt{2}/2, \sqrt{2}/2)$. Thus the tangent line is $\sqrt{2}/2(1 - t, 1 + t)$.

Example 6.14. Now let's consider the curve given by $\vec{r}(t) = (t^2, t^3, 2t)$. We compute the derivative is $\vec{r}\,'(t) = 2t\vec{i} + 3t^2\vec{j} + \vec{2}k$.

If we want to find the tangent line at $t = 2$, we compute $\vec{r}\,'(2) = (4, 12, 2)$, and thus we get an equation for the line $\vec{r}(2) + t\vec{r}\,'(2) = (4, 8, 4) + t(4, 12, 2)$.



After taking the first derivative, we can also take the second (and further) derivatives. As in the single variable case, if the function gives position, and the derivative gives velocity, then the second derivative gives acceleration.

Definition 6.15. The *acceleration* of an object that moves along a path with position $\vec{r}(t)$ at time t is

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = \frac{d^2\vec{r}}{dt^2} = \lim_{h \to 0} \frac{\vec{r}\,'(t+h) - \vec{r}\,'(t)}{h}.$$

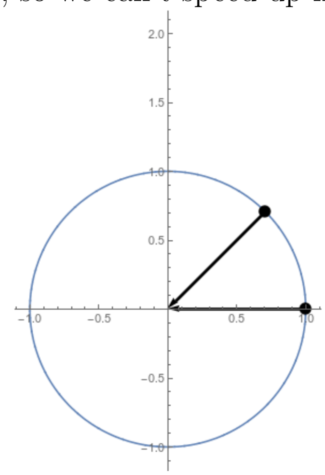
As you'd expect, we can compute the acceleration just by taking the componentwise second derivatives: we have

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = (x''(t), y''(t), z''(t)).$$

Example 6.16. Consider again the circle parametrized by $\vec{r}(t) = (\cos(t), \sin(t))$. Then we know that $\vec{r}\,'(t) = (-\sin(t), \cos(t))$, and thus the second derivative is $\vec{r}\,''(t) = (-\cos(t), -\sin(t))$.

Then we compute that $\vec{r}\,''(0) = (-1, 0)$ and $\vec{r}\,''(\pi/4) = (-\sqrt{2}/2, -\sqrt{2}/2)$.

We notice that the acceleration arrows in a circle always point inwards! This is because the motion is at a constant speed, so we can't speed up in the direction of our velocity.



Example 6.17. Suppose we have the function $\vec{r}(t) = (2, 6, 0) + (t^3 + t)(4, 3, 1)$. Then we can compute the velocity to be $\vec{r}\,'(t) = (3t^2 + 1)(4, 3, 1)$, and the acceleration is given by $\vec{r}\,''(t) = 6t(4, 3, 1)$.

6.2 Surfaces

A curve is a one-dimensional object, and a surface is a two-dimensional object. That means that where a curve needs to be described by one parameter, a surface requires two.

Definition 6.18. A parametrization of a surface is a function $\vec{r} : \mathbb{R}^2 \to \mathbb{R}^n$.

Sometimes write in components: $\vec{r}(s,t) = (x(s,t), y(s,t), z(s,t))$. Each component is a multivariable function $\mathbb{R}^2 \to \mathbb{R}$.

We've already seen some very important examples.

Example 6.19. The graph of a function $z = f(x,y)$ is given by the parametrization $\vec{r}(s,t) = (s,t,f(s,t))$.

Parametrizations and coordinate systems are the same idea—describing a point with a collection of numbers. Thus the alternate coordinate systems we've seen can be viewed as parametrizations.

Example 6.20. We can parametrize a sphere using spherical coordinates. A sphere of radius 5 is parametrized by $\vec{r}(\theta,\phi) = (5\sin\phi\cos\theta, 5\sin\phi\sin\theta, 5\cos\phi)$.

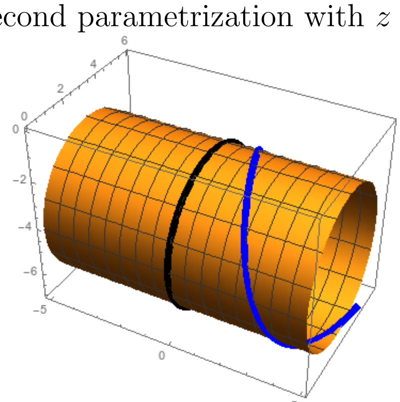
Example 6.21. Let's parametrize a cylinder of radius 1, centered at the origin. We can do this, effectively, with cylindrical coordinates. We have the parametrization $\vec{r}(\theta,z) = (\cos(\theta), \sin(\theta), z)$.

If we want to parametrize a cylinder of radius 3 centered at the line $y = 3, z = -4$, then we just need to tweak this. Notice that this cylinder is pointing in a new direction! We get $\vec{r}(\theta,s) = (s, 3\cos(\theta) + 3, 3\sin(\theta) - 4)$.

As a final remark, we can see that parametrizations aren't unique. Obviously we could instead do something like $\vec{r}(\theta,s) = (s, 3\sin(\theta) + 3, 3\cos(\theta) - 4)$, which would just have the circles oriented in the opposite direction.

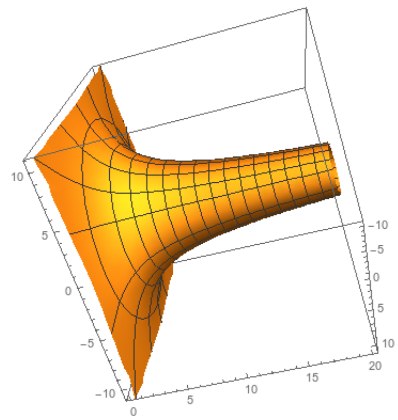
But we could also do something like $\vec{r}(\theta,s) = (s + \theta, 3\sin(\theta) + 3, 3\cos(\theta) - 4)$. This parametrizes exactly the same cylinder! But in the previous parametrizations, holding s constant gives a circle parallel to the yz plane. In this parametrization, holding s constant gives a helix, as in example 6.4.

In the following diagram, the black curve is the image of the first parametrization with $z = 0$. The blue curve is the second parametrization with $z = 0$.



Example 6.22. Let's parametrize a trumpet shape narrowing from a bell at the origin along the direction of the y axis. The radius, as a function of y, is given by $f(x) = \frac{10}{\sqrt{x}}$.

Then we can parametrize the surface by $\vec{r}(y, \theta) = \left(\frac{10\cos(\theta)}{\sqrt{y}}, y, \frac{10\sin(\theta)}{\sqrt{y}}\right)$.



If we fix one parameter on a surface then we get a curve. This is the same idea as level sets and contours that we discussed in section 1.2.

Example 6.23. Consider the cylinder parametrized by $\vec{r}(t, z) = (\cos(t), \sin(t), z)$. The two parameter curves through the point $(0, 1, 1)$ are given by $\vec{r}(t, 1)$, which is a circle of radius 1 in the $z = 1$ plane; and $\vec{r}(\pi/2, z)$, which is a vertical line through the point $(0, 1, 0)$.

In general every parameter curve will be either a circle (if we fix the second parameter) or a vertical line (if we fix the first parameter).

6.3 Change of Coordinates in Integrals

We've mentioned already that some curves and surfaces can be parametrized in more than one way.

Example 6.24. The unit sphere centered at the origin can be parametrized:

$$\vec{r}(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$
$$\vec{r}(\theta, \phi) = (\cos\phi, \sin\phi\cos\theta, \sin\phi\sin\theta)$$
$$\vec{r}(\theta, z) = (\sqrt{1-z^2}\cos\theta, \sqrt{1-z^2}\sin\theta, z)$$
$$\vec{r}(x, y) = (x, y, \sqrt{1-x^2-y^2}).$$

In sections 5.3 and 5.4 we saw that a good choice of coordinate system will often make integrals much easier. But a choice of coordinate system is really just a parametrization.

Example 6.25. Let's parametrize the xy plane. We can parametrize this:

$$\begin{aligned}
\vec{r}(s,t) &= (s,t) \\
\vec{r}(s,t) &= (t,s) \\
\vec{r}(s,t) &= (3s, s-t) \\
\vec{r}(s,t) &= (s\cos t, s\sin t).
\end{aligned}$$

This last parametrization is just polar coordinates.

Remark 6.26. This is the same idea as "change of basis" in a linear algebra context.

In general, we can use customized parametrizations to make double integrals easier. If we have a parametrization given by $\vec{r}(s,t) = (x(s,t), y(s,t))$, this means we'll divide our region up into rectangles in (s,t) coordinates, and sum up the values in each rectangle. To make this useful, we need to figure out how area in (s,t)-coordinates relates to area in (x,y) coordinates.

Suppose we have a a small rectangle in (s,t) coordinates, with corners at

$$(s,t), (s+\Delta s, t), (s, t+\Delta t), (s+\Delta s, t+\Delta t).$$

Then its image under the $\vec{r}$ transformation is going to be *some* four-sided shape with curved sides, with corners at the points

$$(x(s,t), y(s,t)), (x(s+\Delta s, t), y(s+\Delta s, t)), (x(s, t+\Delta t), y(s, t+\Delta t)), (x(s+\Delta s, t+\Delta t), y(s+\Delta s, t+\Delta t)).$$

When $\Delta s, \Delta t$ are small, we can treat this as a parallelogram. So we just need to find the area of a parallelogram.

You might recall from section 2.4 proposition 2.39 that the area of a parallelogram with sides given by the vectors $\vec{u}$ and $\vec{v}$ is $\|\vec{u} \times \vec{v}\|$. So we need to figure out the vectors for the sides of this parallelogram.

But since these vectors are just $\frac{\Delta x}{\Delta s}\vec{i} + \frac{\Delta y}{\Delta s}\vec{j}$ and $\frac{\Delta x}{\Delta t}\vec{i} + \frac{\Delta y}{\Delta t}\vec{j}$, we can approximate these with directional derivatives. In particular, when the sides are small, they are approximately given by $\frac{\partial x}{\partial s}\Delta s\vec{i} + \frac{\partial y}{\partial s}\vec{j}$ and $\frac{\partial x}{\partial t}\Delta t\vec{i} + \frac{\partial y}{\partial t}\Delta t\vec{j}$. Thus the area of the parallelogram is approximately

$$\begin{aligned}
\left| \left(\frac{\partial x}{\partial s}\Delta s\vec{i} + \frac{\partial y}{\partial s}\vec{j} \right) \times \left(\frac{\partial x}{\partial t}\Delta t\vec{i} + \frac{\partial y}{\partial t}\Delta t\vec{j} \right) \right| &= \left| \frac{\partial x}{\partial s}\Delta s \frac{\partial y}{\partial t}\Delta t - \frac{\partial x}{\partial t}\Delta t \frac{\partial y}{\partial s}\Delta s \right| \\
&= \left| \frac{\partial x}{\partial s}\frac{\partial y}{\partial d} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} \right| \Delta s \Delta t.
\end{aligned}$$

Definition 6.27. We define the *Jacobian* of a function to be the determinant of the matrix of partial derivatives. Thus the Jacobian of $\vec{r} : \mathbb{R}^2 \to \mathbb{R}^2$ is

$$\frac{\partial(x,y)}{\partial(s,t)} = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix}.$$

Thus the area of the parallelogram we're studying is approximately $\left|\frac{\partial(x,y)}{\partial(s,t)}\right| \Delta s \Delta t$.

So now let's return to thinking about our integral. If we want to compute an integral, we have the following computation:

$$\begin{aligned}\int_r f(x,y)\,dA &= \lim \sum f(u_{ij}^*, v_{ij}^*) \left|\frac{\partial(x,y)}{\partial(s,t)}\right| \Delta s \Delta t \\ &= \lim \sum f(x(s_{ij}^*, t_{ij}^*), y(s_{ij}^*, t_{ij}^*)) \left|\frac{\partial(x,y)}{\partial(s,t)}\right| \Delta s \Delta t \\ &= \int_T f(x(s,t), y(s,t)) \left|\frac{\partial(x,y)}{\partial(s,t)}\right| ds\,dt.\end{aligned}$$

Thus in summary, we can compute integrals in a new coordinate system by doing the following:

1. Substitue $x(s,t)$ and $y(s,t)$ for x and y in the inside of the integral.
2. Change the region/bounds to be described in terms of s and t.
3. Make the substitution $dx\,dy = \left|\frac{\partial(x,y)}{\partial(s,t)}\right| ds\,dt$.

Remark 6.28. This is a generalization of u-substitution in single-variable calculus. Recall there that if $x = g(u)$ then

$$\int_a^b f(g(u))g'(u)\,du = \int_{g(a)}^{g(b)} f(x)\,dx.$$

Just as in this reparametrization, we do a substitution inside the variable; we change the bounds; and we have a correcting factor from the derivative of $x = g(u)$.

Example 6.29. We can recover polar coordinates from this setup. Polar coordinates give the parametrization $x = r\cos\theta, y = r\sin\theta$. Then we compute

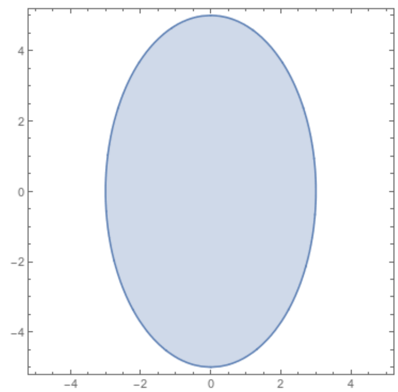
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

This gives us back exactly the conversion factor that we got before.

Example 6.30. Let's find the area of an ellipse given by the equation $x^2/a^2 + y^2/b^2 = 1$.

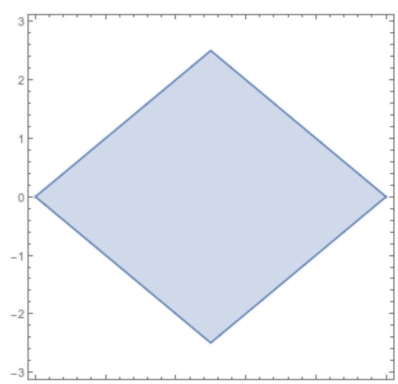
We can parametrize this with $x = as, y = bt$. Then the equation becomes $s^2 + t^2 = 1$, so the region is the unit circle in the st plane. We calculate the Jacobian is $\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$. Thus the area of the ellipse is

$$\int_R 1 \, dx \, dy = \int_T 1ab \, ds \, dt = ab \int_T 1 \, ds \, dt = ab\pi.$$



Example 6.31. Evaluate $\int_R x+y \, dA$ where R is the region with vertices $(0,0), (5,0), (5/2, 5/2)$, and $(5/2, -5/2)$.

We can calculate that the equations for the boundary lines are $y = x, y = -x, y = x-5$, and $y = 5-x$.



There are two basic ways we could approach this. One is to set up a pair of double integrals with x, y coordinates: we get

$$I = \int_0^{5/2} \int_{-x}^{x} x + y \, dy \, dx + \int_{5/2}^{5} \int_{x-5}^{5-x} x + y \, dy \, dx.$$

But this is long and annoying.

The other thing we can do is use a change of coordinates to convert this into a reasonable rectangle. We see that the region isn't particularly aligned in the directions of $\vec{i}$ and $\vec{j}$, but rather in the directions $\vec{i} + \vec{j}$ and $\vec{i} - \vec{j}$. So we might try a parametrization $x = s + t$ and $y = s - t$.

To find our new bounds we plug this into our boundary equations. For $y = x$ we get $s - t = s + t$, which gives us $t = 0$. For $y = -x$ we get $s - t = -s - t$, which gives us $s = 0$.

Similarly, for $y = x - 5$ we get $s - t = s + t - 5$. Solving gives $t = 5/2$. Finally, we have $y = 5 - x$, which gives $s - t = 5 - s - t$, which gives $s = 5/2$.

Thus, rather than having a complicated integral setup, we just get bounds $0 \leq s \leq 5/2, 0 \leq t \leq 5/2$. Our integrand is $x + y = s + t + s - t = 2s$. And our Jacobian is

$$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = |-1-1| = 2.$$

Thus we have the integral

$$\begin{aligned} I &= \int_0^{5/2} \int_0^{5/2} 2s \cdot 2 \, dt \, ds \\ &= \int_0^{5/2} 4st|_0^{5/2} \, ds = \int_0^{5/2} 10s \, ds \\ &= 5s^2|_0^{5/2} = \frac{125}{4}. \end{aligned}$$

Example 6.32. We can also generalize this to three variables. Spherical coordinates are given by the transformation $x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho\cos\phi$. Then we compute the Jacobian is

$$\begin{aligned} \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} &= \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} \\ &= |-\rho^2\sin^3\phi\cos^2\theta - \rho^2\sin^2\theta\sin\phi\cos^2\phi \\ &\qquad - \rho^2\sin\phi\cos^2\phi\cos^2\theta - \rho^2\sin^3\phi\sin^2\theta| \\ &= \rho^2 \left|\sin^3\phi(\cos^2\theta + \sin^2\theta) + \sin\phi\cos^2\phi(\sin^2\theta + \cos^2\theta)\right| \\ &= \rho^2 \left|\sin^2\phi(\sin\phi + \cos^2\phi)\right| \\ &= \rho^2|\sin\phi| = \rho^2\sin\phi. \end{aligned}$$

(We can drop the absolute values around sin because $\sin\phi \geq 0$ when $\phi \in [0, \pi]$).

6.4 Vector Fields

In subsection 6.1 we talked about curves, which are functions $\vec{r} : \mathbb{R} \to \mathbb{R}^3$. In subsection 6.2 we talked about surfaces, which are functions $\vec{r} : \mathbb{R}^2 \to \mathbb{R}^3$. Now we'll discuss vector fields, which are functions $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$. (Or, more generally, functions $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$).

The basic idea is that sometimes we have a flow, or a force field, or a current. What these all have in common is that for every point they have a direction and a magnitude, which represents the flow of the current, or the force of the force field. Thus we want to write a function that takes in a location, which is a point in $\mathbb{R}^3$, and outputs a vector in $\mathbb{R}^3$.

Example 6.33.

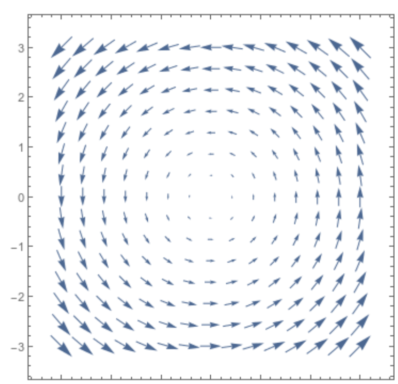
- The current direction and distance you have to go to reach your destination.
- The force exerted by the Earth's gravitic force.
- The direction and speed of the currents in a river.

Definition 6.34. A *vector field* in $\mathbb{R}^n$ is a function $\mathbb{F} : \mathbb{R}^n \to \mathbb{R}^n$ that takes in a point in $\mathbb{R}^n$ and outputs a vector in $\mathbb{R}^n$.

Example 6.35. Consider the 2-dimensional vector field given by $\vec{F}(x, y) = -y\vec{i} + x\vec{j}$. We can get a sense for what this looks like by plotting a few points.

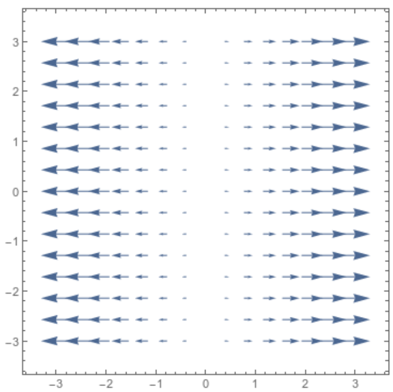
We see that $\vec{F}(x, 0) = x\vec{j}$, so along the x-axis the arrows point straight up, and get longer the further away from the origin we are. Similarly, we have $\vec{F}(0, y) = -y\vec{i}$, so the arrows point to the left and get longer the further we are from the origin. And we see that, e.g. $\vec{F}(1, 1) = -\vec{i} + \vec{j}$ points up and left.

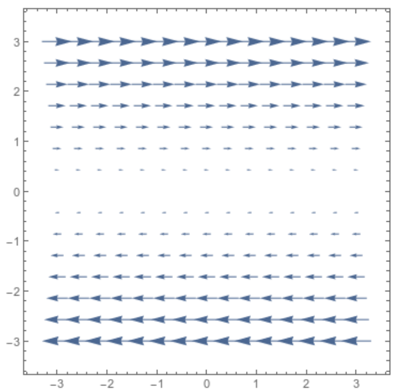
This seems to be a roughly counterclockwise circular motion. And indeed, we get the plot:



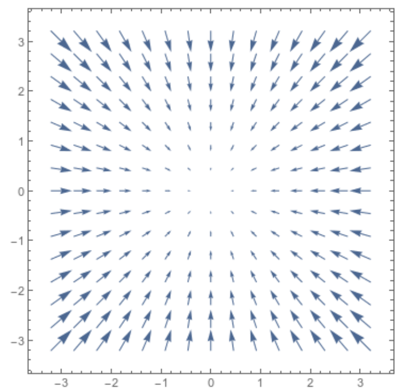
Example 6.36. Consider $\vec{F}(x, y) = x\vec{i}$. This vector field is just arrows that always point horizontally, with their size determined by their x-coodrinate.

Now consider $\vec{G}(x, y) = y\vec{i}$. This is still horizontal arrows, but now their length is determined by their y-coordinate.



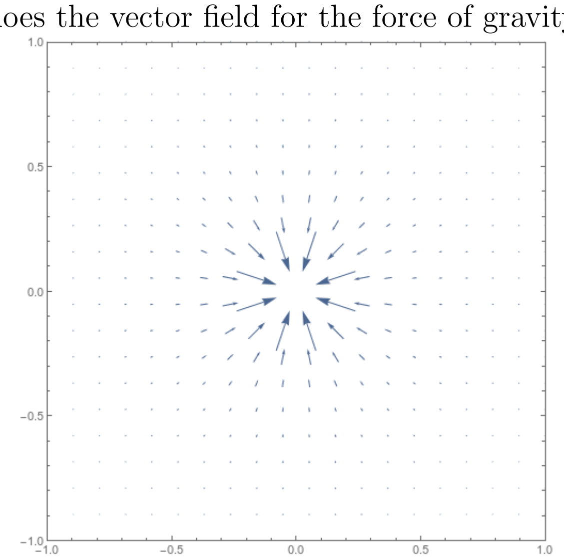


Example 6.37. What's an equation for this vector field?



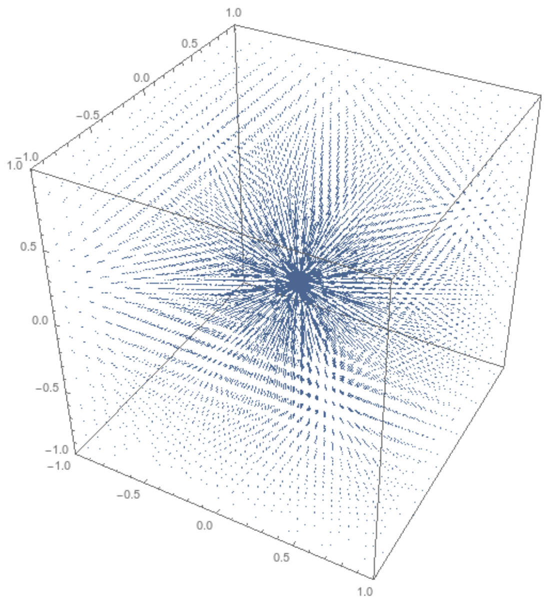
We see that the arrows are always pointing back towards the origin. This vector field has the function $\vec{F}(x, y) = -x\vec{i} - y\vec{j}$.

Example 6.38. What does the vector field for the force of gravity look like?



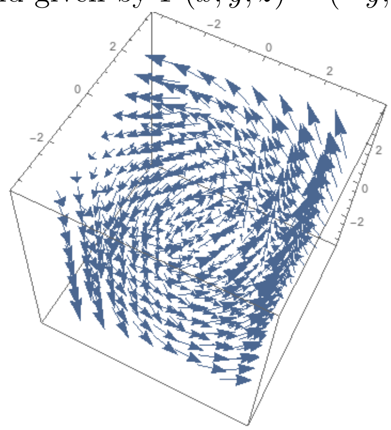
We can also think about vector fields in three dimensions, although the pictures are much harder to draw.

Example 6.39. What does this gravity field look like in three dimensions? We get



The formula here is
$$\vec{F}(\vec{r}) = \frac{GMm}{\|\vec{r}\|^2}\left(\frac{-\vec{r}}{\|\vec{r}\|}\right) = \frac{-GMm\vec{r}}{\|\vec{r}\|^3}.$$

Example 6.40. The vector field given by $\vec{F}(x, y, z) = (-y, x, z)$ is

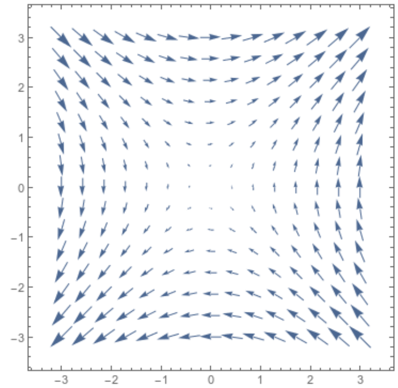


It's hard to make out from the picture, but this an upwards-twisting cylinder.

A common and important source of vector fields is the gradient function. Notice that if $f : \mathbb{R}^3 \to \mathbb{R}$ is a multivariable function, then $\nabla f = (f_x, f_y, f_z)$ is a function from $\mathbb{R}^3 \to \mathbb{R}^3$, and thus is a vector field.

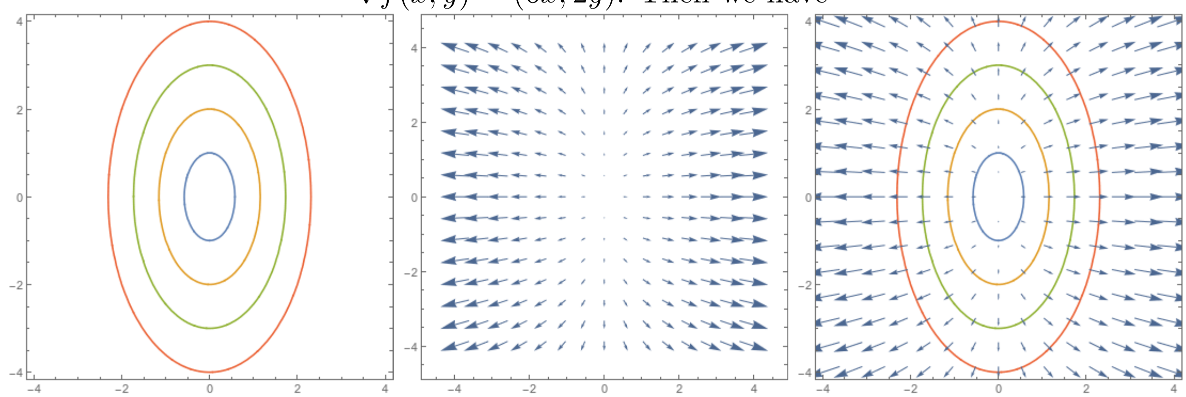
This in fact makes sense, since the gradient tells you, for every point, what the direction and magnitude of greatest increase is. Thus gradients often give us vector fields.

Example 6.41. Let $f(x, y) = xy$. Then $\nabla f(x, y) = (y, x)$ has the plot



We can infer the vector plot of a function from its contour diagram, since we know the gradient vectors are always perpendicular to the contours.

Example 6.42. Consider the function $f(x, y) = 3x^2 + y^2$. We calculate that $\nabla f(x, y) = (6x, 2y)$. Then we have



Another way to visualize vector fields is through *flow lines*. If we think of a vector field as describing the direction you will move from a given point, then flow lines tell us the path we will follow if we start at a given point.

Definition 6.43. Let $\mathbb{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, and $\vec{r} : \mathbb{R} \to \mathbb{R}^n$ be a curve. We say that $\vec{r}$ is a *flow line* of $\vec{F}$ if $\vec{r}\,'(t) = \vec{F}(\vec{r}(t))$. that is, the velocity of the path is equal to the vector field.

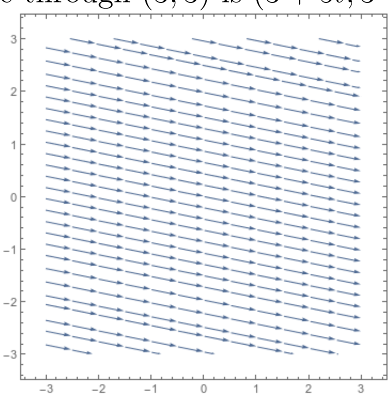
In a flow line, the vector field describes the motion of a particle along the curve.

The *flow* of a vector field is the collection of all the flow lines.

Remark 6.44. We can view this as a solution to the differential equation $\vec{r}\,'(t) = \vec{F}(\vec{r}(t))$. The choice of flow line is determined by your initial conditions.

Example 6.45. Let $\vec{v}(t) = 5\vec{i} - \vec{j}$. What does a flow line look at? What is an equation for the flow line that goes through $(3, 3)$?

A flow line would just be a line in the direction $5\vec{i} - \vec{j}$, and thus a line that looks like $\vec{r}_0 + t(5, -1)$. Then the flow line through $(3, 3)$ is $(3 + 5t, 3 - t)$.



Example 6.46. Let $\vec{F}(x, y) = y\vec{i} + \vec{j}$. Find the path of an object in the flow that is at the point $(2, 2)$ at time $t = 0$.

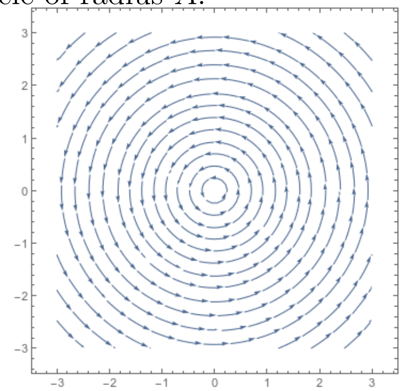
We know that $\vec{r}\,'(t) = (y, 1)$, so we have $x'(t) = y(t)$ and $y'(t) = 1$. The second equation tells us that $y(t) = t + y_0$. Then the first equation tells us that since $x'(t) = t + y_0$ we have $x(t) = t^2/2 + y_0 t + x_0$.

Plugging in our initial conditions tells us that $x_0 = 2$ and $y_0 = 2$. Thus we have $\vec{r}(t) = (t^2/2 + 2t + 2, t + 2)$.

Example 6.47. Let $\vec{F}(x, y) = -y\vec{i} + x\vec{j}$. What does the flow of this vector field look like?

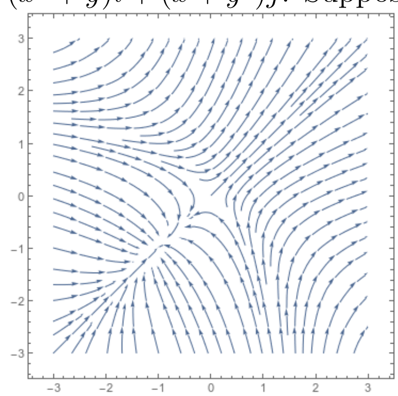
We have $x'(t) = -y(t)$ and $y'(t) = x(t)$. Thus in particular we have $x''(t) = -y(t)$, and this tells us that $x(t) = A\cos(t) + B\sin(t)$. Taking the case $B = 0$, we then have $x(t) =$

$A\cos(t)$ and then we can see that $y(t) = A\sin(t)$. Then we have $\vec{r}(t) = (A\cos(t), A\sin(t))$ and thus each flow line is a circle of radius A.



We can (often) find flow lines exactly by solving a system of differential equations. (If you want to learn more about this you should take Math 340: Ordinary Differential Equations). But often solving them exactly is annoying or impractical, and we want to approximate the flow lines instead. We can find these by Euler's Method.

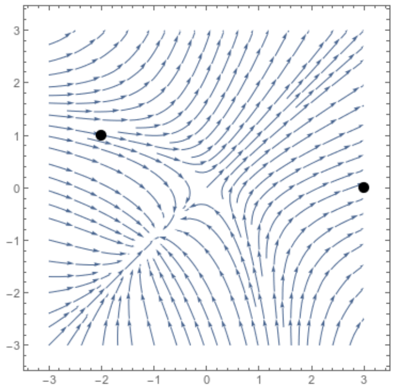
Example 6.48. Let $\vec{F}(x,y) = (x^2+y)\vec{i} + (x+y^2)\vec{j}$. Suppose $\vec{r}(0) = (-2,1)$. What is $\vec{r}(1)$?



We can estimate this with a linear approximation. If we want a very coarse estimate, we can simply calculate

$$\begin{aligned} \vec{r}\,'(0) &= \vec{F}(-2,1) = 5\vec{i} - \vec{j} \\ \vec{r}(1) &\approx \vec{r}(0) + \vec{r}\,'(0)(1-0) = (-2,1) + (5,-1) = (3,0). \end{aligned}$$

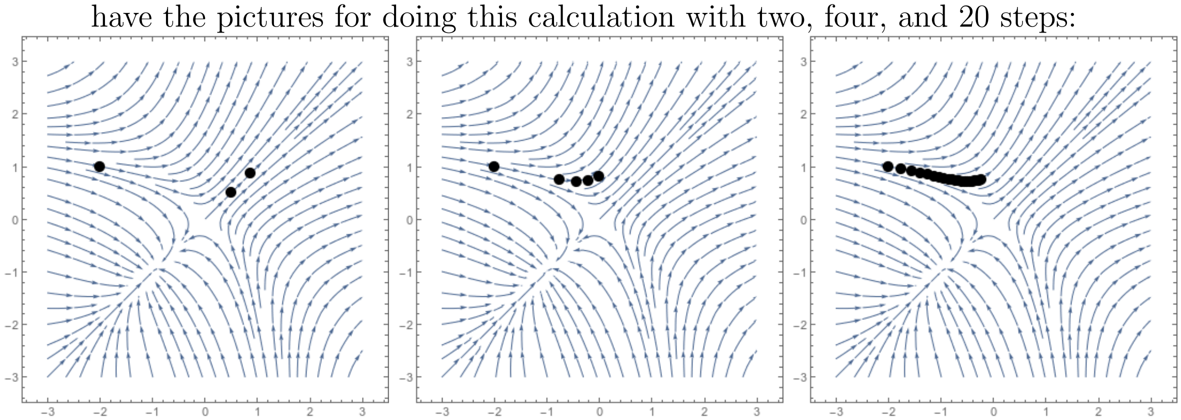
But from the picture below we see that this isn't really a very good estimate. Basically, our Δt is much too big, so our linear approximation isn't very good. (We can see this in the graph because the flow lines curve).



If we knew the second derivatives, we could fix this. But we can also fix this just by making smaller step sizes. We can compute

$$
\begin{aligned}
\vec{r}'(0) &= \vec{F}(-2,1) = 5\vec{i} - \vec{j} \\
\vec{r}(1/2) &\approx \vec{r}(0) + \vec{r}'(0)(1/2-0) = (-2,1) + \frac{1}{2}(5,-1) = (1/2,1/2) \\
\vec{r}(1) &\approx \vec{r}(1/2) + \vec{r}'(1/2)(1-1/2) = (1/2,-1/2) + \frac{1}{2}(3/4,3/4) = (7/8,7/8)
\end{aligned}
$$

and we see this is much close to the "true" answer. We can get the answer as close as we want to the true answer by taking more and more smaller and smaller steps. Below we have the pictures for doing this calculation with two, four, and 20 steps:



7 Line Integrals

From this point on, our course will essentially have two topics:

1. Different types of integrals you might want to compute for one reason or another; and
2. Ways to avoid actually having to compute those integrals.

In this section we will discuss various types of one-dimensional integrals—integrals over curves. In the next section we'll discuss two-dimensional integrals, and in the final section of the course we'll discuss some important theorems that relate one-, two-, and three-dimensional integrals.

7.1 Integrating over a curve

In single-variable calculus, we studied single-variable integrals. Recall the basic idea here is that we have some function defined on real numbers, and we want to "add up" all the values our function has over some interval in the real numbers.

We want to generalize this concept to multiple dimensions somehow. We may have some multivariable function defined on $\mathbb{R}^2$ or $\mathbb{R}^3$, and we want to add up all the values of it we will see.

In section 5 we talked about adding up all the values of a 2-variable function in a 2-dimensional region. But sometimes we'll only take a *path* through that region, and we want to add up all the values on that path.

Example 7.1.

- If the energy it takes to move is a function of location in space, then the total energy used will be an integral of that function over the path traveled.
- If the density of a wire is a function of the point in space, then the total mass will be the integral of density over the length of the wire.
- The total length of a curve will be the integral of the function 1 over the length of the curve.

How do we compute a line integral? Let's start by working out the length of a curve. We can do this by approximating the curve with a large number of short (tangent) lines, and then adding up all their lengths. We see that as the lines get shorter and more numerous, the approximation gets better.

Suppose we have a curve parametrized by $\vec{r}(t) = (x(t), y(t), z(t))$, for $a \leq t \leq b$.

$$L \approx \sum_{i=1}^{n} \|\vec{r}(t_{i+1}) - \vec{r}(t_i)\| = \sum_{i=1}^{n} \left\| \frac{\vec{r}(t_{i+1}) - \vec{r}(t_i)}{\Delta t} \right\| \Delta t.$$

As the number of line segments tends to infinity, the quotient in the middle tends to $\|\vec{r}\,'(t_n)\|$, and we so see that this sum is

$$L \approx \sum_{i=1}^{n} \|\vec{r}\,'(t_i)\| \Delta t \approx \int_a^b \|\vec{r}\,'(t)\| \, dt.$$

Proposition 7.2. *If C is a curve with parametric equation $\vec{r}(t)$ for $a \leq t \leq b$, then the length of the curve is given by*

$$\int_a^b \|\vec{r}\,'(t)\| \, dt.$$

Example 7.3. Consider the ellipse $(2\cos(t), \sin(t))$ for $0 \leq t \leq 2\pi$. What is the circumference?

We compute

$$\int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2} = \int_0^{2\pi} \sqrt{4\sin^2(t) + \cos^2(t)} \approx 9.69.$$

So what was "really" going on there? If we want to find the size of a shape—the length of an interval, the area of a region, the volume of a region—we can integrate the function 1 over that shape. So we just want to integrate the function 1 over the curve $C = \vec{r}(t)$.

So how do integrals like this work in general? For a single-variable integral, we chopped the interval up into a bunch of subintervals. To integrate over a curve, we want to chop the curve up into a bunch of short lines. Then we evaluate the function at a point on each short subline, multiply by the length of that subline, and add all these things up.

But we just worked out that the length of a subline was

$$\|\vec{r}(t_{n+1}) - \vec{r}(t_n)\| = \left\| \frac{\vec{r}(t_{n+1}) - \vec{r}(t_n)}{\Delta t} \right\| \Delta t \approx \|pvecr'(t_n)\| \Delta t.$$

Thus our Riemann sum is approximately given by

$$I = \sum_{i=1}^{n} f(\vec{r}(t_n)) \|\vec{r}\,'(t_n)\| \Delta t \approx \int_a^b f(\vec{r}(t)) \, dt.$$

Thus we define:

Definition 7.4. The *scalar line integral* of a multivariable function $f(\vec{r})$ over a curve $C = \vec{r}(t)$ for $a \leq t \leq b$ is

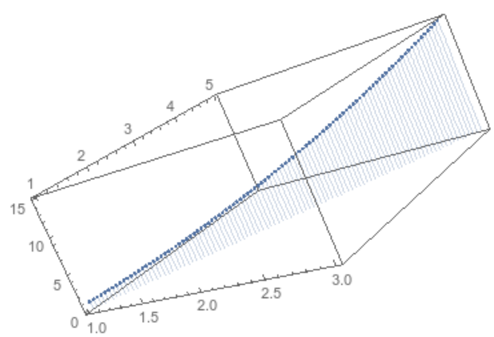
$$\int_C f(\vec{r}) \, ds = \lim_{\|\Delta \vec{r}_i\| \to 0} \sum_{i=1}^{n} \vec{F}(\vec{r}_i) \|\Delta r_i\| = \int_a^b f(\vec{r}(t)) \|\vec{r}\,'(t)\| \, dt.$$

Remark 7.5. The $\|\vec{r}'(t)\|$ term is playing the same role here that the Jacobian did in our reparametrized two-variable integrals. The Jacobian measured how much our parametrization stretched the area of a surface parametrization; this term is measuring how much our parametrization stretches the length of a curve.

Example 7.6. If C is the line from $(1,1)$ to $(3,5)$, find $\int_C xy\,ds$.

We parametrize C by $\vec{r}(t) = (1+2t, 1+4t)$ for $0 \leq t \leq 1$. Then we have

$$\begin{aligned} f(\vec{r}(t)) &= (1+2t)(1+4t) \\ \|\vec{r}'(t)\| &= \|(2,4)\| = 2\sqrt{5} \int_C xy\,ds \\ &= \int_0^1 (1+2t)(1+4t)2\sqrt{5}\,dt = 2\sqrt{5}\int_0^1 1+6t+8t^2\,dt \\ &= 2\sqrt{5}(t+3t^2+8t^3/3)|_0^1 = 2\sqrt{5}(1+3+8/3) = \frac{40\sqrt{5}}{3}. \end{aligned}$$



Example 7.7. Suppose we have a helical spring lying along a path parametrized by $\vec{r}(t) = (\cos t, \sin t, t)$ for $0 \leq t \leq 2\pi$, and suppose the density of the wire is given by $1+z$. What is the total mass of the wire?

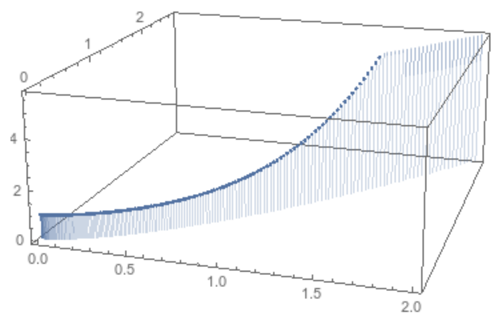
We have

$$\begin{aligned} f(\vec{r}(t)) &= 1 + (t\cos t) \\ \|\vec{r}'(t)\| &= \|(-\sin(t), \cos(t), 1)\| = |\sin^2(t) + \cos^2(t) + 1| = \sqrt{2} \\ \int_C f(\vec{r})\,ds &= \int_0^{2\pi} (1+t)\sqrt{2}\,dt \\ &= \sqrt{2}(t+t^2/2)|_0^{2\pi} = \sqrt{2}(2\pi+2\pi^2) = 2\pi\sqrt{2}(1+\pi). \end{aligned}$$

Example 7.8. Suppose we have a wire lying along the path given by $\vec{r}(t) = (t^2/2, t^3/3)$ for $0 \leq t \leq 2$, with density given by $1+xy^2$. What is the total mass of the wire?

We compute

$$\begin{aligned} f(\vec{r}(t)) &= 1 + t^8/18 \\ \|\vec{r}\,'(t)\| &= \|(t, t^2)\| = t\sqrt{1+t^2} \\ \int_C f(\vec{r})\, ds &= \int_0^2 (1 + t^8/18) t\sqrt{1+t^2}\, dt \approx 15.2. \end{aligned}$$



Remark 7.9. You might worry that this isn't well defined. A wire in space could be parametrized by a large number of different functions $\vec{r}(t)$. However, we will get the same answer for any parametrization! Changing the parametrization will change the output of $f(\vec{r}(t))$, but it will also change $\|\vec{r}'(t)\|$, and the two changes will exactly cancel out.

This is another way of thinking of the role of the derivative term in these integrals. It makes sure every bit of length is counted equally, even if your parametrization moves over some parts of the curve quickly and moves over other parts slowly.

7.2 Integrating vector fields over a curve

In the past section, we saw how to take some scalar quantity that varies with position, and add up its value over some extended curve through space. However, it's probably more common to worry about a *field* that extends through space.

Example 7.10.

- The work done by a particle moving through a force field along some path
- The extent to which a fluid current or circulation points in the same direction as a path

What these phenomena has in common is that we want to measure the extent to which a vector field points in the same direction as our curve.

If the vector field is constant and the path is a straight line, both of these computations are easy. Work is equal to the distance traveled times the force in the direction of the distance: thus work is equal to the *dot product* of force and displacement. We might write that $W = \vec{F} \cdot d\vec{r}$.

Similarly, if we want to know the extent to which a current is flowing in the direction of some line $\vec{r}$, we compute $\vec{F} \cdot \vec{r}$. Both of these notions involve the dot product. But this computation only works when the vector field is constant, and the path is a straight line.

So let's do our usual integral approximation trick. We parametrize our curve $C = \vec{r}(t)$ for $a \leq t \leq b$, and break it up into a bunch of line segments. Then for each line segment, the direction is constant and the vector field is approximately constant, so we have work equal to

$$W_i = \vec{F}(\vec{r}(t_i)) \cdot (\vec{r}(t_{i+1}) - \vec{r}(t_i)).$$

Thus the work done over the entire path is approximately

$$\begin{aligned} W &\approx \sum_{i=1}^{n} \vec{F}(\vec{r}(t_i)) \cdot (\vec{r}(t_{i+1}) - \vec{r}(t_i)) \\ &= \sum_{i=1}^{n} \vec{F}(\vec{r}(t_i)) \cdot \frac{\vec{r}(t_{i+1}) - \vec{r}(t_i)}{\Delta t} \Delta t \\ &\approx \sum_{i=1}^{n} \vec{F}(\vec{r}(t_i)) \cdot \vec{r}\,'(t_i) \Delta t \\ &\approx \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt. \end{aligned}$$

Definition 7.11. We define the *line integral of a vector field* $\vec{F}$ over a curve $C = \vec{r}(t)$ to be

$$\int_C \vec{F} \cdot d\vec{r} = \lim_{\|\Delta \vec{r}_i\| \to 0} \sum_{i=1}^{n} \vec{F}(\vec{r}_i) \cdot \Delta \vec{r}_i = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt.$$

Remark 7.12. As before, the choice of parametrization doesn't affect the result of this integral. Proving this rigorously is incredibly annoying so we won't bother.

Example 7.13. Let $\vec{F}(x, y) = \vec{i} + \vec{j}$ and let C be a curve going in a straight line from $(1, 1)$ to $(4, 1)$. What is $\int_C \vec{F} \cdot d\vec{r}$?

We don't actually need to set up an integral here, since the field is constant and the curve is a straight line. We have

$$\int_C \vec{F} \cdot d\vec{r} = (\vec{i} + \vec{j}) \cdot (3, 0) = 3.$$

If we do set up an integral, we have $\vec{r}(t) = (1 + t, 1)$ for $0 \leq t \leq 3$. Then our integral is

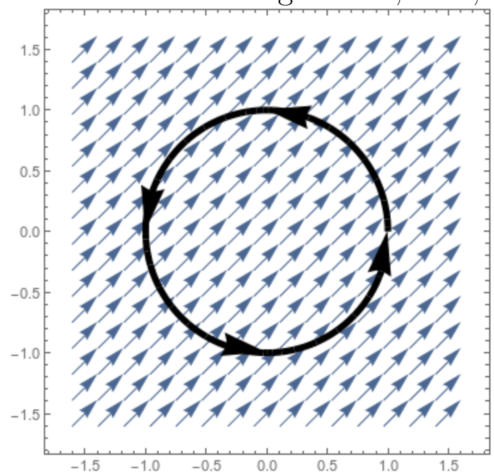
$$\int_C \vec{F} \cdot d\vec{r} = \int_0^3 (\vec{i} + \vec{j}) \cdot (\vec{i})\, dt = \int_0^3 1\, dt = 3.$$

Example 7.14. If C is the circular path centered at the origin that begins and ends at $(1, 0)$, oriented counterclockwise, and $\vec{F}(x, y) = \vec{i} + \vec{j}$, what is $\int_C \vec{F} \cdot d\vec{r}$?

We can parametrize the unit circle by $\vec{r}(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$. Then we have

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} (\vec{i} + \vec{j}) \cdot (-\sin(t)\vec{i} + \cos(t)\vec{j})\, dt \\ &= \int_0^{2\pi} \cos t - \sin t\, dt \\ &= \sin(t) + \cos(t)|_0^{2\pi} = 0 + 1 - (0 + 1) = 0. \end{aligned}$$

We could have predicted this result from the picture. We see that the current is flowing *against* the circle exactly as often as it is flowing *with* it; thus, the net is zero.



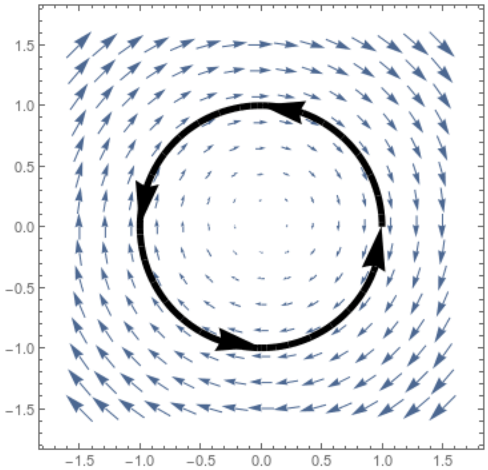
If we integrate over only the first half of the circle, we'd expect to get something negative. And in fact we do:

$$\begin{aligned}\int_C \vec{F} \cdot\, d\vec{r} &= \int_0^{\pi} (\vec{i} + \vec{j}) \cdot (-\sin(t)\vec{i} + \cos(t)\vec{j})\, dt \\ &= \int_0^{\pi} \cos t - \sin t\, dt \\ &= \sin(t) + \cos(t)|_0^{\pi} = 0 - 1 - (0+1) = -2.\end{aligned}$$

Example 7.15. Let $\vec{F}(x,y) = (y, -x)$ and let C be the unit circle parametrized to go counterclockwise, as in example 7.14. Then

$$\begin{aligned}\int_C \vec{F} \cdot\, d\vec{R} &= \int_0^{2\pi} (\sin t\vec{i} - \cos t\vec{j}) \cdot (-\sin t, \cos t)\, dt \\ &= \int_0^{2\pi} -\sin^2 t - \cos^2 t\, dt = \int_0^{2\pi} -1\, dt = -2\pi.\end{aligned}$$

Again from the picture we can tell that this integral should be negative, since the circle is running against the direction of the flow.

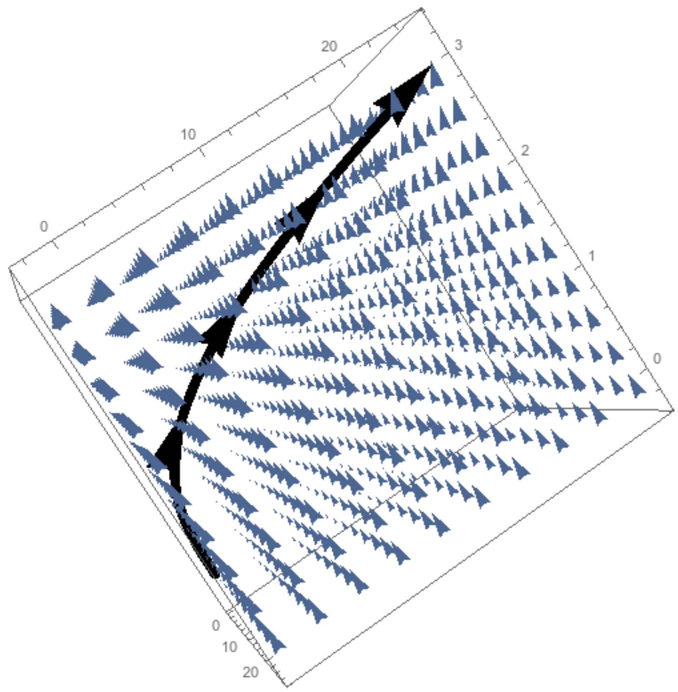


What if we parametrize the circle going the other way? Geometrically, we see that our dot product will reverse, so we'd expect to get the opposite of what we got originally. In fact that's exactly what happens: we parametrize our curve as $(\cos(t), -\sin(t))$ for $0 \leq t \leq 2\pi$ and we get

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{R} &= \int_0^{2\pi} (-\sin t\vec{i} - \cos t\vec{j}) \cdot (-\sin t, -\cos t)\, dt \\ &= \int_0^{2\pi} \sin^2 t + \cos^2 t\, dt = \int_0^{2\pi} 1\, dt = 2\pi.\end{aligned}$$

Example 7.16. Suppose a particle moves along the path $\vec{r}(t) = (t, 3t^2, t^3-2)$ through a force field $\vec{F}(x, y, z) = y\vec{i} + x^2\vec{j} + xz\vec{k}$. What is the total work done over the interval $0 \leq t \leq 3$?

$$\begin{aligned}\int_C \vec{F}\, d\vec{r} &= \int_0^3 (3t^2, t^2, t^4 - 2t) \cdot (1, 6t, 3t^2)\, dt \\ &= \int_0^3 3t^2 + 6t^3 + 3t^6 - 6t^3\, dt \\ &= t^3 + 3t^7/7|_0^3 = 27 + 3^8/7 = \frac{6750}{7}.\end{aligned}$$



We sometimes write our line integrals in what's known as "differential" notation. If we have $\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$, and our curve is parametrized by $\vec{r}(t) = (x(t), y(t), z(t))$, then we will sometimes write

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz.$$

In this context, dx is just $\frac{dx}{dt}dt = x'(t)\,dt$.

Example 7.17. If C is a line segment from $(0,0)$ to $(3,4)$, evaluate

$$\int_C y\,dx + xy\,dy.$$

We can parametrize C by $\vec{r}(t) = (3t, 4t)$ for $0 \leq t \leq 1$. Then we have

$$\int_c y\,dx + xy\,dy = \int_0^1 4t \cdot 3\,dt + 12t^2 \cdot 4\,dt = \int_0^1 12t + 48t^2\,dt = 6t^2 + 16t^3|_0^1 = 22.$$

Notice we haven't changed anything geometrically here, and we haven't changed any of the computations you're doing; we're just writing things out differently.

Finally, here's a bit of a trolly example.

Example 7.18. Let $\vec{F}(x,y) = x\vec{i} + y\vec{j}$ be a field of force exerted by some force field, and suppose we have a wire lying along the path $\vec{r}(t) = (2t, 3t)$ for $0 \leq t \leq 2$. What is the total force on the wire?

Since we're totaling the forces along the length of the wire, we expect to compute an integral. However, we're not asking for work but for total force, so we don't need to take a dot product. Instead we would have to compute a "scalar" integral of the vector field, and get

$$\begin{aligned}
\vec{F} &= \int_0^2 (2t\vec{i} + 3t\vec{j})(t\sqrt{13})\,dt = \sqrt{13}\int_0^2 2t^2\vec{i} + 3t^2\vec{j}\,dt \\
&= \sqrt{13}\left(2t^3/3\vec{i} + t^3\vec{j}\right)\Big|_0^2 \\
&= \sqrt{13}\left(16/3\vec{i} + 8\vec{j}\right) = \frac{16\sqrt{13}}{3}\vec{i} + 8\sqrt{13}\vec{j}.
\end{aligned}$$

Notice that our output here is a vector, as it should be since we're asking for the force exerted on an object.

7.3 Conservative Vector Fields

Example 7.19. Let's compute the integral of $\vec{F}(x,y) = y\vec{i} + x\vec{j}$ along the straight line from $(0,0)$ to $(1,1)$. We can parametrize this by $\vec{r}(t) = (t,t)$ for $0 \leq t \leq 1$, and we have

$$\int_0^1 (t,t)\cdot(1,1)\,dt = \int_0^1 t + t\,dt = \int_0^1 2t\,dt = t^2|_0^1 = 1.$$

But what if we went between the same two points in a different way? We could follow a path going to the right, and then up. Then we would have $\vec{r}_1(t) = (t,0)$ and $\vec{r}_2(t) = (1,t)$, each from 0 to 1. The total line integral would be

$$\int_0^1 (0,t)\cdot(1,0)\,dt + \int_0^1 (t,1)\cdot(0,1)\,dt = \int_0^1 0\,dt + \int_0^1 1\,dt = 1.$$

Or we could follow a circular path, going clockwise around the circle centered at $(1,0)$. We'd parametrize this with $\vec{r}(t) = (\cos(t)+1, -\sin(t))$ for $\pi \leq t \leq 3\pi/2$, and then we'd have

$$\begin{aligned}
\int_\pi^{3\pi/2} (-\sin(t), \cos(t)+1)\cdot(-\sin(t), -\cos(t))\,dt &= \int_\pi^{3\pi/2} \sin^2(t) - \cos^2(t) - \cos(t)\,dt \\
&= -\sin(t) - \frac{1}{2}\sin(2t)|_\pi^{3\pi/2} \\
&= -\sin(3\pi/2) - \frac{1}{2}\sin(3\pi) + \sin(\pi) + \frac{1}{2}\sin(2\pi) \\
&= 1 - 0 + 0 + 0 = 1.
\end{aligned}$$

This is the same thing we got before! This isn't an accident.

Definition 7.20. We say that a vector field $\vec{F}$ is *conservative* or *path-independent* if whenever C_1 and C_2 are two curves with the same starting point and the same end point, then $\int_{C_1} \vec{F} \cdot d\vec{R} = \int_{C2} \vec{F} \cdot d\vec{r}$.

We say a curve is *closed* if it has the same starting point and ending point. A vector field is conservative if whenever C is a closed curve, then $\int_C \vec{F} \cdot d\vec{r} = 0$.

Remark 7.21. We call these fields conservative because in physics, they represent a field that follows some sort of conservation of energy law.

Visually, we can tell whether a vector field is conservative by seeing whether we can draw a closed curve with positive line integral. For instance, we see that the field from example 7.14 is conservative, since in any closed curve the work will exactly balance out and we'll wind up with a zero integral. In contrast, the vector field from example 7.15 is clearly not conservative, since we integrated a closed circle over it and got an answer of 2π.

If we know we have a conservative vector field, we can dramatically simplify some integrals.

Example 7.22. Compute the integral done by the force field $\vec{F}(x, y) = y\vec{i} + x\vec{j}$, which we know to be conservative, on a particle following the path given by $\vec{r}(t) = (t+\sin^8(\pi t)\cos^5(t), \cos^4(2\pi t)+2t)$ for $0 \leq t \leq 1$.

Obviously we don't want to actually do anything with that parametric path. However, we see that $\vec{r}(0) = (0, 1)$ and $\vec{r}(1) = (1, 3)$. Since $\vec{F}$ is path independent, we can just compute the integral over the path $\vec{r}_1(t) = t, 1 + 2t$. And we have

$$\int_0^1 (1 + 2t, t) \cdot (1, 2)\, dt = \int_0^1 1 + 4t\, dt = t + 2t^2|_0^1 = 3.$$

But we can actually make our job even easier after a little bit of work and a clever observation.

We first observe that if $\vec{F} = \nabla f$ for some multivariable function f, then $\vec{F}$ is conservative. In particular, we prove the following:

Proposition 7.23 (Fundamental Theorem of Calculus for Line Integrals)**.** *If C is a piecewise smooth oriented path from the point P to the point Q, and f is some function that is continuously differentiable on the path C, then*

$$\int_C \nabla f \cdot d\vec{r} = f(Q) - f(P).$$

In particular, this implies that any gradient field is conservative, since the line integral depends only on the starting and ending points of the curve.

Remark 7.24.

1. This theorem often allows us to avoid having to compute a line integral at all.
2. This theorem is the analogue to the single-variable Fundamental Theorem of Calculus: in some sense, f is the antiderivative of ∇f.
3. If $\nabla f = \vec{F}$, we sometimes say that f is a *potential function* for $\vec{F}$. For instance, if $\vec{F} = \nabla f$ is the gravitic force field, then f measures your gravitational potential energy.

Proof. We can view $f(Q) - f(P)$ as the change in f as the input moves from Q to P. We know from linear approximation that $f(Q) \approx f(P) + \nabla f(P) \cdot (Q - P)$. But this approximation won't be very good if P and Q are far apart.

We can improve the approximation by dividing the path along the curve up into a bunch of short line segments. Then the total change in the value of f along the whole path is exactly $f(Q) - f(P)$, but it is approximately the sum of the approximate change along each of these line segments. Thus we have

$$\begin{aligned} f(Q) - f(P) &\approx \sum_{i=1}^{n} \nabla f(\vec{r}(t_i)) \cdot (\vec{r}(t_{i+1}) - \vec{r}(t)) \\ &= \sum_{i=1}^{n} \nabla f(\vec{r}(t_i)) \cdot \frac{\vec{r}(t_{i+1}) - \vec{r}(t_i)}{\Delta t} \Delta t \\ &\approx \sum_{i=1}^{n} \nabla f(\vec{r}(t_i)) \cdot \vec{r}\,'(t_i) \Delta t \approx \int_C \nabla f \cdot d\vec{r}. \end{aligned}$$

□

Example 7.25. If we look back at example 7.19 that began this subsection, we may observe that $\vec{F}(x, y) = y\vec{i} + x\vec{j} = \nabla(xy)$. Thus for each problem, we were computing $f(1, 1) - f(0, 0) = 1 \cdot 1 - 0 \cdot 0 = 1$.

In example 7.22 we computed $f(1, 3) - f(0, 1) = 1 \cdot 3 - 0 \cdot 1 = 3 - 0 = 3$.

Example 7.26. If $f(x, y) = x^2 y - y^2$, and C is a spiral that begins at $(1, 2)$ and ends at $(3, 1)$, we compute

$$\int_C \nabla f \cdot d\vec{r} = f(3, 1) - f(1, 2) = 8 - (-2) = 10.$$

We just showed that every gradient field is conservative. It turns out that the converse is also true: every conservative vector field is the gradient of some function. In fact, we can be very (uselessly) specific about this:

Proposition 7.27. *Let $\vec{F}$ be a conservative vector field. Then we can define a function f by picking any point P, and define*

$$f(Q) = \int_C \vec{F} \cdot d\vec{r}$$

where C is any curve that begins at P and ends at Q. Then $\vec{F} = \nabla f$.

Remark 7.28. Recall that this is related to the fundamental theorem of calculus for single variable integrals. There, we said that if f is a continuous function, it has an antiderivative $F(x) = \int_a^x f(t)\,dt$.

We can get a number of different potential functions here, depending on our choice of starting point P. This choice corresponds to the choice of a in the single-variable case, which corresponds to the choice of constant in the $+C$ of the antiderivative.

Thus a vector field $\vec{F}$ is conservative if and only if it is the gradient of some potential function f.

Example 7.29. Find a potential function for $\vec{F}(x, y) = y \cos x\vec{i} + (\sin x + y)\vec{j}$.

Suppose $\vec{F}$ has a potential function. Then we have some function $f(x, y)$ such that $f_x(x, y) = y \cos x$ and $f_y(x, y) = \sin x + y$.

The first equation tells us that $f(x) = y \sin x + g(y)$ for some g. The second equation tells us that $f(x, y) = y \sin(x) + y^2/2 + h(x)$ for some h. Combining these two facts gives us $f(x) = y \sin(x) + y^2/2 + C$ for some constant C.

Example 7.30. Is $\vec{F}(x, y) = 2y\vec{i} + x\vec{j}$ conservative?

Suppose $\nabla f(x, y) = \vec{F}(x, y)$. Then we have $f_x(x, y) = 2y$ and $f_y(x, y) = x$. The first equation tells us that $f(x, y) = 2xy + g(y)$, and the second equation tells us that $f(x, y) = xy + h(x)$.

These two equations are incompatible, and thus $\vec{F}$ isn't the gradient of any potential function, and thus is not conservative.

This gives us a hint for how to figure out if a vector field is conservative. If $\vec{F}$ is conservative, then we have $\vec{F}(x, y) = f_x(x, y)\vec{i} + f_y(x, y)\vec{j}$. In particular, if we take the y derivative of the first term and the x derivative of the second term, we will get the mixed partial in either case.

That is, if we have some f such that $f_x = 2y$, then $f_{xy} = 2$. But if $f_y = x$, then $f_{xy} = 1$, which is a contradiction.

Thus we see that if $\vec{F}(x, y) = F_1(x, y)\vec{i} + F_2(x, y)\vec{j}$, then $\vec{F}$ is conservative if and only if $\frac{\partial F_1}{\partial x} = \frac{\partial F_2}{\partial y}$. Generalizing this statement leads into our next topic of discussion.

7.4 The Curl of a Vector Field

Let's step back to consider the geometry of a vector field for a bit. We've been considering vector fields that are conservative, which means the integral over any closed loop is zero. What does it look like for a vector field to be non-conservative?

A non-conservative vector field will have some closed loops where the line integral is non-zero. Thus, the vector field will have some component that looks like it is rotating. We can compute "how much" the vector field is rotating around some point by taking the line integral of a circle centered at that point; as the circle gets smaller, we ignore everything except that point and get a number that represents the rotation there.

Definition 7.31. The *circulation density* of a vector field at a point (x, y, z) around the unit vector $\vec{n}$ is

$$\operatorname{circ}_{\vec{n}} \vec{F}(x, y, z) = \lim_{\text{area}\to 0} \frac{\int_C \vec{F} \cdot d\vec{r}}{\text{area of } C}$$

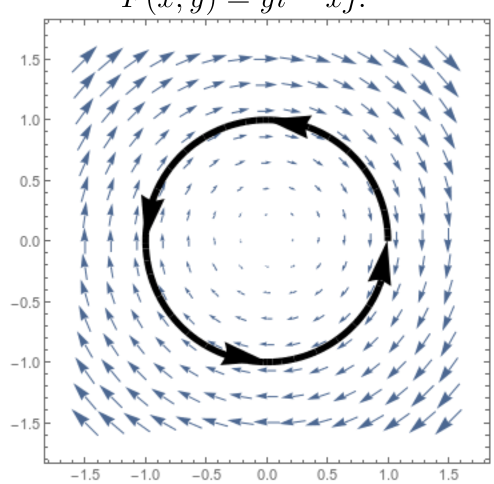
Where C is a circle perpendicular to $\vec{n}$ oriented by the right-hand rule.

Remark 7.32. Recall that we describe rotation with a vector perpendicular to the plane of rotation, according to the right-hand rule. So rotation that appears clockwise to us is represented by a vector pointing away from us; rotation that appears counterclocwise is represented by a vector pointing towards us.

Thus if $\vec{n}$ is pointing towards us, $\operatorname{circ}_{\vec{n}}$ measures the circulation in the direction we would identify as counterclockwise.

Example 7.33. Let's look at the circulation density at the origin of the vector field

$$\vec{F}(x, y) = y\vec{i} - x\vec{j}.$$



If a circle centered at the origin has radius a then we can parametrize it by $\vec{r}(t) =$

$(a\cos t, a\sin t)$. Then our line integral is

$$\int_0^{2\pi} (a\sin(t), -a\cos(t)) \cdot (-a\sin(t), a\cos(t))\, dt = \int_0^{2\pi} -a^2\, dt = -2\pi a^2.$$

The area of the circle is of course πa^2, so the circulation density is

$$\lim_{a\to 0} \frac{-2\pi a^2}{\pi a^2} = -2.$$

What does this tell us? It tells us that at the origin, the vector field is rotating counterclockwise with magnitude -2—or, in other words, it's rotating clockwise with magnitude 2. Which is exactly what we see from the picture.

Now, this process makes perfect sense geometrically, but is not fun to compute with. Fortunately, there is a much easier way to deal with this.

Definition 7.34. Let $\vec{F}(x,y,z) = F_1(x,y,z)\vec{i} + F_2(x,y,z)\vec{j} + F_3(x,y,z)\vec{k}$ be a vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$. We define the *curl* of $\vec{F}$ to be

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}.$$

If $\vec{F}$ is a vector field from $\mathbb{R}^2 \to \mathbb{R}^2$, this reduces to

$$\nabla \times F = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

and we often treat it as a scalar quantity.

How is the curl related to the circulation density?

- The direction of $\nabla \times \vec{F}$ is the direction $\vec{n}$ which maximizes $\operatorname{circ}_{\vec{n}} \vec{F}$.
- The magnitude of $\nabla \times \vec{F}$ is the circulation density in that direction.

Thus the curl tells you in which direction, if any, a vector field is rotating at any given point.

Example 7.35. Looking back at the field $\vec{F}(x,y) = y\vec{i} - x\vec{j}$ of example 7.33, we can compute the curl:

$$\nabla \times \vec{F} = \frac{\partial - x}{\partial x} - \frac{\partial y}{\partial y} = -1 - -1 = -2.$$

Thus not only is the circulation -2 at the origin; it is in fact -2 everywhere.

We saw in section 7.3 that a conservative 2-dimensional vector field will have zero curl. We can actually make this statement much stronger:

Proposition 7.36. *Let $\vec{F}$ be a vector field. If $\vec{F}$ is conservative, then $\nabla \times F = \vec{0}$.*

If $\vec{F}$ is also defined everywhere, and $\nabla \times \vec{F} = \vec{0}$, then $\vec{F}$ is conservative.

That is, we can almost say that a vector field is conservative if and only if its curl is zero. However, if a vector field has a singularity, it may have zero curl and still not be conservative.

Example 7.37. Is $\vec{F}(x, y) = 2xy\vec{i} + xy\vec{j}$ conservative?

We compute

$$\nabla \vec{F} = \frac{\partial xy}{\partial x} - \frac{\partial 2xy}{\partial y} = y - 2x \neq 0.$$

Thus $\vec{F}$ is not conservative.

Example 7.38. Is $\vec{F}(x, y, z) = yz\vec{i} + xz\vec{j} + xy\vec{k}$ conservative?

$$\nabla \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x - x)\vec{i} + (y - y)\vec{j} + (z - z)\vec{k} = \vec{0}.$$

Thus $\vec{F}$ is conservative. In fact, we see that $\vec{F}(x, y, z) = \nabla xyz$.

Example 7.39. Is $\vec{F}(x, y, z) = y\vec{i} + z\vec{j} + x\vec{k}$ conservative?

$$\nabla \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = (0 - 1)\vec{i} + (0 - 1)\vec{j} + (0 - 1)\vec{k} = -\vec{i} - \vec{j} - \vec{k} \neq \vec{0}.$$

Thus this vector field is not conservative.

In contrast, we see that $\vec{F}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$ is conservative. It is the gradient of $x^2/2 + y^2/2 + z^2/2$.

However, remember that if $\vec{F}$ isn't defined everywhere, the curl test does not work.

Example 7.40. Let $\vec{F}(x, y) = \frac{-y\vec{i} + x\vec{j}}{x^2 + y^2}$. This function is undefined, and in fact has an infinite singularity, at the origin.

We calculate that

$$\begin{aligned}
\nabla \times \vec{F}(x,y) &= \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \\
&= \frac{1(x^2+y^2) - (2x)x}{(x^2+y^2)^2} - \frac{(-1)(x^2+y^2) + y(2y)}{(x^2+y^2)^2} \\
&= \frac{y^2 - x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2} = 0.
\end{aligned}$$

However, we can also take the line integral around the unit circle. We parametrize $\vec{r}(t) = (\cos(t), \sin(t))$ and calculate

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \frac{1}{\cos^2(t) + \sin^2(t)} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t))\, dt \\
&= \int_0^{2\pi} \sin^2(t) + \cos^2(t)\, dt = \int_0^{2\pi} dt = 2\pi.
\end{aligned}$$

(We also could have noticed, geometrically, that this vector field is always tangent to the unit circle, and thus the integral is equal to the magnitude of the vector field times the length of the curve, which is $1 \cdot 2\pi$).

Thus the curl of $\vec{F}$ is zero, but $\vec{F}$ is not conservative, since it is undefined at zero. This idea is actually very important. In complex analysis it gives us the idea of *residues*, which are used in a large number of computations; my research uses the residue of a particular type of function around an infinite singularity at zero to give information about number theory.

In geometry and topology it gives us the idea of *(de Rham) cohomology*, which reverses the idea: if we don't know where our domain has holes, we can compute the integral of a zero-curl vector field over a closed loop and see if it is equal to zero.

7.5 Green's Theorem

We have now seen that the curl of a vector field measures, in some sense, the value of a small line integral. It seems like we should then be able to relate the curl to a line integral—if the curl at each point gives the value of a line integral at that point, then perhaps adding up many values of the curl corresponds to adding up many line integrals? In fact this is precisely the case.

Theorem 7.41 (Green). *Suppose C is a piecewise smooth simple closed curve that is the boundary of a region R in the plane, and oriented so that the region is to the left as we move* *around the curve, and suppose $\vec{F} = F_1\vec{i} + F_2\vec{j}$ is a smooth vector field on an open region containing R and C. Then*

$$\int_C \vec{F} \cdot d\vec{r} = \int_R \left(\nabla \times \vec{F}(x,y)\right) \cdot \vec{k}\, dx\, dy.$$

Remark 7.42. We sometimes write, equivalently, that

$$\int_C F_1\, dx + F_2\, dy = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy.$$

These pieces are interchangeable: the left-hand sides are different ways of writing the same line integral, and the right-hand sides are different ways of writing the same two-dimensional integral.

Proof. The basic idea of the proof is that the curl gives the counterclockwise line integral at every point, so taking the integral of the curl will give you the sum of a lot of little line integrals, which add up into one big line integral.

First we'll prove the theorem for a rectangular region. Consider the region $R = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$, and let $\vec{F}(x,y)$ be defined on this region. Let C be the path counterclockwise around the outside of this rectangle, with C_1 going from (a,c) to (a,d), C_2 from (a,d) to (b,d), and so on.

Then we can see easily that

$$\begin{aligned}
\int_{C_1} \vec{F} \cdot d\vec{r} &= \int_a^b (F_1(x,c), F_2(x,c)) \cdot (1,0)\, dx = \int_a^b F_1(x,c)\, dx \\
\int_{C_3} \vec{F} \cdot d\vec{r} &= \int_a^b -F_1(x,d)\, dx \\
\int_{C_1+C_3} \vec{F} \cdot d\vec{r} &= \int_a^b F_1(x,c) - F_1(x,d)\, dx
\end{aligned}$$

But $F_1(x,c) - F_1(x,d) = \int_d^c \frac{\partial F_1}{\partial y}\, dy$ by the Fundamental Theorem of Calculus, so after switching the order of the bounds of integration, we have

$$\int_{C_1+C_3} F \cdot d\vec{r} = \int_a^b \int_c^d -\frac{\partial F_1}{\partial y}\, dy\, dx.$$

We can use a similar argument to work out that

$$\int_{C_2+C_4} F \cdot d\vec{r} = \int_c^d \int_a^b \frac{\partial F_2}{\partial x}\, dx.$$

Adding these together gives the integral around the whole rectangle:

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1+C_2+C_3+C_4} \vec{F} \cdot d\vec{r} = \int_a^b \int_c^d \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dy \, dx.$$

To prove this for an arbitrary region: we divide the region up into a large number of small rectangles. On each rectangle, the theorem is true. The integral over the whole region is approximately the sum of the integrals of the small regions (with an error term coming from the fact that the rectangles don't cover the region exactly).

Less obviously, the integral over the boundary of R is approximately the sum of the integrals of the boundaries of each rectangle: two adjacent rectangles will share a boundary segment, but oriented in opposite ways so they cancel out. After adding the boundaries of all the rectangles together, everything will cancel except the segments on the boundary (which again only approximate the boundary).

Thus we have

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &\approx \sum_i \int_{C_i} \vec{F} \cdot d\vec{r} \\ &= \sum_i \int_{R_i} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dy \, dx \\ &\approx \int_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dy \, dx.\end{aligned}$$

In the limit as we take infinitely many rectangles, these approximations become equalities.

□

Remark 7.43. Green's Theorem only works if $\vec{F}$ is defined on the entire region R. If $\vec{F}$ is undefined somewhere, then the integral over the region is undefined, and so Green's Theorem doesn't make sense.

Green's theorem is a really useful tool for converting otherwise difficult integrals into much easier ones.

Example 7.44. Let $\vec{F}(x, y) = y\vec{i} + x^2\vec{j}$, and let C be the counterclockwise path around the rectangle R described by $0 \leq x \leq 3, 1 \leq y \leq 2$. Compute $\int_C \vec{F} \cdot d\vec{r}$.

This would be really annoying to do directly: we'd have to do four different integrals over the line segments comprising the perimeter of R. Fortunately, Green's Theorem makes our life easier.

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dx \, dy \\ &= \int_1^2 \int_0^3 2x - 1 \, dx \, dy = \int_1^2 x^2 - x|_0^3 \, dy = \int_1^2 6 \, dy = 6.\end{aligned}$$

Example 7.45. Let's integrate $(yx^2 - y)\,dx + (x^3 + 4)\,dy$ over a path that goes from $(0, 0)$ to $(3, 2)$, then to $(3, 0)$, then back to $(0, 0)$.

We could parametrize each segment individually, but that seems annoying. Instead, we use Green's theorem. We have

$$\begin{aligned}
\frac{\partial F_2}{\partial x} &= \frac{\partial x^3 + 4}{\partial x} = 3x^2 \\
\frac{\partial F_1}{\partial y} &= \frac{\partial yx^2 - y}{\partial y} = x^2 - 1 \\
\int_C \vec{F} \cdot d\vec{r} &= \int_R 3x^2 - (x^2 - 1)\,dA = \int_0^3 \int_0^{2x/3} 2x^2 + 1\,dy\,dx \\
&= \int_0^3 4x^3/3 + 2x/3\,dx = 3^4/3 + 3^2/3 = 30.
\end{aligned}$$

Except we did one thing subtly wrong. Recall the curve needs to be oriented so the region is on the left; but the curve we want the line integral of has the region on the right. Thus we get the opposite of what we want, and the true answer is -30.

Less often, we use Green's Theorem to go in the other direction, and replace a double integral with a line integral. One fun example is using Green's Theorem to compute area.

Recall that the area of a region is $\int_R 1\,dA$. Is there a vector field such that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$? In fact there are several; common choices include $x\vec{j}$, $-y\vec{i}$, and $-y/2\vec{i} + x/2\vec{j}$. Then by Green's Theorem we have

$$A = \int_C x\,dy = -\int_C y\,dx = \frac{1}{2}\int x\,dy - y\,dx$$

where C is any curve that traverses the boundary of the region, keeping the region on the left.

Example 7.46. Let's use this to find the area of a circle of radius a. We can parametrize the circle with $\vec{r}(t) = (a\cos t, a\sin t)$. Then if we integrate $x\vec{j}$ around the outside, we have

$$A = \int_C x\,dy = \int_0^{2\pi} (a\cos t) \cdot a\cos t\,dt = a^2 \int_0^{2\pi} \cos^2(t)\,dt.$$

We could do that integral, knowing the antiderivative of cosine, but it's annoying. So instead we use the more complicated-looking vector field $(-y/2\vec{i} + x/2\vec{j})$:

$$\begin{aligned}
A &= \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} a\cos(t)a\cos(t) - a\sin(t)(-a\sin t)\,dt \\
&= \frac{a^2}{2}\int_0^{2\pi} \cos^2(t) + \sin^2(t)\,dt = \frac{a^2}{2}\int_0^{2\pi} 1\,dt = a^2\pi.
\end{aligned}$$

8 Surface Integrals

Want to do the same thing for surfaces we did for curves.

8.1 Scalar surface integrals

In section 7.1 we looked at integrating a function over a curve, where we added up the value of the function at all points on that curve. We used this to find average values and to find total mass of a wire from its density.

Often we have a 2-dimensional object or "surface" that we want to do the same adding-up process for. In this section we'll see how to do this.

As usual, we want to break our region up into rectangles, evaluate the function on each rectangle, multiply by the area of the rectangle, and then add them all up. So how do we do this?

If our surface were a region in the plane, we'd already know. So describe with a region in the plane. This is exactly what a parametrization does!

Suppose our surface is parametrized by $\vec{r}(s,t)$ for $a \leq s \leq b, c \leq t \leq d$. We can certainly divide the st rectangle into a bunch of subrectangles. This corresponds on the surface to a bunch of (approximate) parallelograms. So we want to multiply the value of the function on each parallelogram by the area of each parallelogram.

For a given parallelogram, the value of the function f is going to be $f(\vec{r}(s,t))$. So we just need to find the area of the parallelogram.

Recall from section 2.4 that the area of a parallelogram is the magnitude of the cross product of the two sides, $\|\vec{u} \times \vec{v}\|$. In 6.3 we used this to work out the area of a parallelogram parametrized by $\vec{r}(s,t)$. We saw that the sides were

$$\frac{\Delta x}{\Delta s}\vec{i} + \frac{\Delta y}{\Delta s}\vec{j} + \frac{\Delta z}{\Delta s}\vec{k} \approx \frac{\partial x}{\partial s}\Delta s\vec{i} + \frac{\partial y}{\partial s}\Delta s\vec{j} + \frac{\partial z}{\partial s}\Delta s\vec{k} = \vec{r}_s(s,t)\Delta s$$
$$\frac{\Delta x}{\Delta t}\vec{i} + \frac{\Delta y}{\Delta t}\vec{j} + \frac{\Delta x}{\Delta t}\vec{k} \approx \frac{\partial x}{\partial t}\Delta t + \frac{\partial y}{\partial t}\Delta t\vec{j} + \frac{\partial z}{\partial t}\Delta t\vec{k} = \vec{r}_t(s,t)\Delta t$$

so the area of the parallelogram is

$$\|\vec{r}_s(s,t)\Delta s \times \vec{r}_t(s,t)\Delta t\| = \|\vec{r}_s(s,t) \times \vec{r}_t(s,t)\|\Delta s\Delta t.$$

Thus we define

Definition 8.1. The *surface integral* of the function $f(x, y, z)$ on the surface S parametrized by $\vec{r}(s, t)$ over a planar region R is

$$\begin{aligned}\int_S f\,dS &= \lim \sum_{i,j} f(\vec{r}(s_i, t_i))\|\vec{r}_s(s_i, t_i) \times \vec{r}_t(s_i, t_i)\|\Delta s \Delta t \\ &= \int_R f(\vec{r}(s, t))\|\vec{r}_s(s, t) \times \vec{r}_t(s, t)\|\,ds\,dt.\end{aligned}$$

Example 8.2. Integrate $x^2 z$ over cylinder of radius 2, height 3, with base at $z = 0$.

Parametrization: $\vec{r}(\theta, z) = (2\cos\theta, 2\sin\theta, z)$.

$$\begin{aligned}\vec{r}_\theta(\theta, z) &= (-2\sin\theta, 2\cos\theta, 0) \\ \vec{r}_z(\theta, z) &= (0, 0, 1) \\ \vec{r}_\theta \times \vec{r}_z &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (2\cos\theta, 2\sin\theta, 0) \\ \|\vec{r}_\theta \times \vec{r}_z\| &= \sqrt{4\cos^2\theta + 4\sin^2\theta} = \sqrt{4} = 2.\end{aligned}$$

So

$$\begin{aligned}\int_S x^2 z\,ds &= \int_0^{2\pi}\int_0^3 4\cos^2\theta \cdot 2\,dz\,d\theta \\ &= 8\int_0^{2\pi} 3\cos^2\theta\,d\theta = 24\left(\frac{\theta}{2} + \frac{1}{4}\sin(2\theta)\right)\Big|_0^{2\pi} = 24\pi.\end{aligned}$$

Example 8.3. Find the mass of a hemisphere (the half of the sphere with $z \geq 0$ of radius 1 centimeter with density of z grams per square centimeter.

As with all of our integrals, we need to find a parametrization; then we'll compute bounds and the area correction term, and then have a straightforward integral.

We can parametrize the sphere using spherical coordinates with $\rho = 1$. Thus we take

$$\vec{r}(\theta, \phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$$

for $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/2$. We compute the parallelogram area by

$$\begin{aligned}
\vec{r}_\theta(\theta,\phi) &= (-\sin\theta\sin\phi, \cos\theta\sin\phi, 0)\\
\vec{r}_\phi(\theta,\phi) &= (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)\\
\vec{r}_\theta \times \vec{r}_\phi &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta\sin\phi & \cos\theta\sin\phi & 0 \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \end{vmatrix}\\
&= (-\cos\theta\sin^2\phi - 0)\vec{i} + (0 - \sin\theta\sin^2\phi)\vec{j} + (-\sin^2\theta\sin\phi\cos\phi - \cos^2\theta\sin\phi\cos\phi)\vec{k}\\
&= -\cos\theta\sin^2\phi i - \sin\theta\sin^2\phi\vec{j} - \sin\phi\cos\phi\vec{k}\\
\|\vec{r}_\theta \times \vec{r}_\phi\| &= \sqrt{\cos^2\theta\sin^4\phi + \sin^2\theta\sin^4\phi + \sin^2\phi\cos^2\phi}\\
&= \sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi} = \sqrt{\sin^2\phi} = |\sin\phi|
\end{aligned}$$

Since $0 \leq \phi \leq \pi/2$ we know that $\sin\phi \geq 0$ and can drop the absolute values.

Thus our integral is

$$\begin{aligned}
\int_S f\,dS &= \int_0^{2\pi}\int_0^{\pi/2} \cos\phi \cdot \sin\phi\,d\phi\,d\theta\\
&= \int_0^{2\pi} \frac{1}{2}\sin^2\phi|_0^{\pi/2}\,d\theta = \int_0^{2\pi}\frac{1}{2}\,d\theta = \pi.
\end{aligned}$$

Example 8.4. Set up an integral for the surface area of the graph of $z = x^2 - y^2$ over the square $-1 \leq x \leq 1, -1 \leq y \leq 1$.

To find surface area, we need to integrate the function 1 over the surface. Since this surface is a graph, parametrization is easy: we can take $\vec{r}(x,y) = (x, y, x^2 - y^2)$ for $1 \leq x, y \leq 1$. We compute

$$\begin{aligned}
\vec{r}_x &= (1, 0, 2x)\\
\vec{r}_y &= (0, 1, -2y)\\
\vec{r}_x \times \vec{r}_y &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2x \\ 0 & 1 & -2y \end{vmatrix} = -2x\vec{i} - 2y\vec{j} + \vec{k}\\
\|\vec{r}_x \times \vec{r}_y\| &= \sqrt{1 + 4x^2 + 4y^2}.
\end{aligned}$$

Then our integral is

$$\int_{-1}^1\int_{-1}^1 \sqrt{1+4x^2+4y^2}\,dy\,dx = 4 - \frac{1}{3}\arctan(4/3) + \frac{7\ln(5)}{3} \approx 7.45.$$

Working out some of these cross product terms is really annoying. Fortunately, we can precompute a bunch of them so we don't have to do it again.

Proposition 8.5. *If we parametrize a sphere of radius r with*

$$\vec{r}(\theta, \phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi),$$

then

$$\|\vec{r}_\theta \times \vec{r}_\phi\| = r^2 \sin\phi$$

If we parametrize a cylinder of radius r with

$$\vec{r}(\theta, z) = (r\cos\theta, r\sin\theta, z),$$

then

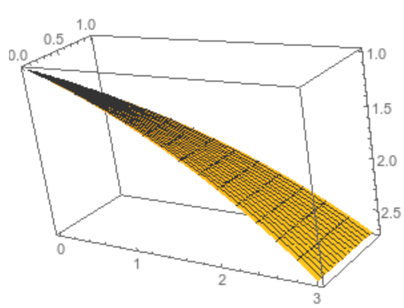
$$\|\vec{r}_\theta \times \vec{r}_z\| = r.$$

If we parametrize the graph of a function $f(x, y)$ with $\vec{r}(x, y) = (x, y, f(x, y))$, then

$$\|\vec{r}_x \times \vec{r}_y\| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}$$

However, we sometimes still need to do surface integrals over non-standard parametrizations.

Example 8.6. Integrate x^2/y over the surface parametrized by $\vec{r}(s, t) = (e^s, st, 3s)$ for $1 \le s, t \le 2$.



We compute

$$\begin{aligned}
\vec{r}_s(s,t) &= (e^s, t, 3) \\
\vec{r}_t(s,t) &= (0, s, 0) \\
\vec{r}_s \times \vec{r}_t &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ e^s & t & 3 \\ 0 & s & 0 \end{vmatrix} = -3s\vec{i} + se^s\vec{k} \\
\|\vec{r}_s \times \vec{r}_t\| &= \sqrt{9s^2 + s^2e^{2s}} = s\sqrt{9 + e^{2s}} \\
\int_S x^2/y \, dS &= \int_1^2 \int_1^2 \frac{e^{2s}}{st} \cdot s\sqrt{9+e^{2s}} \, ds \, dt \\
&= \int_1^2 \frac{1}{3t} \left(9 + e^{2s}\right)^{3/2} |_1^2 \, dt = \left((9+e^4)^{3/2} - (9+e)^{3/2}\right) \int_1^2 \frac{1}{3t} \, dt \\
&= \left((9+e^4)^{3/2} - (9+e)^{3/2}\right) \frac{\ln(t)}{3} |_1^2 \\
&= \left((9+e^4)^{3/2} - (9+e)^{3/2}\right) \frac{\ln(2)}{3}.
\end{aligned}$$

8.2 Flux Integrals

A more common thing we want to do with surface integrals is compute flux of a vector field.

Definition 8.7. The *orientation* of a surface is a continuous choice of normal vector at every point. For a rectangle this just means choosing which side is the "front"; for more complex surfaces it often tells you which side is "up".

The *area vector* of an oriented surface is vector $\vec{A}$ with direction the orientation, and magnitude area of the surface.

The *flux* of a vector $\vec{v}$ through a flat oriented surface is $\vec{v} \cdot \vec{A}$.

Remark 8.8. Not every surface can be given a consistent orientation, but we won't really be worrying about non-orientable surfaces in this course.

Flux measures amount of flow through surface. What if surface isn't flat? Or flow isn't constant? Approximate by a bunch of flat surfaces, flow is locally constant, so can use constant flux. Then add up.

Definition 8.9. The *flux integral* of the vector field $\vec{F}$ through the oriented surface $\vec{S}$ is

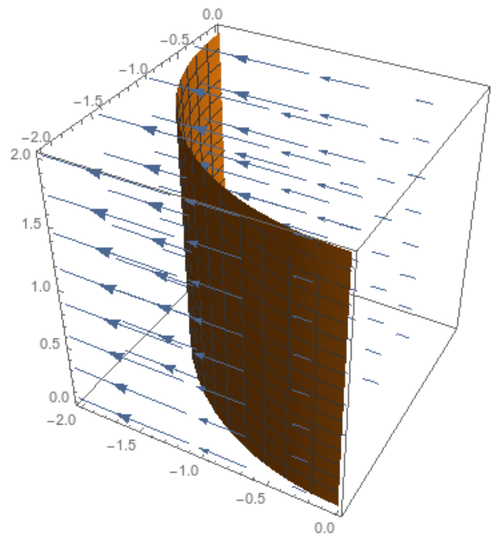
$$\int_S \vec{F} \cdot d\vec{A} = \lim \sum \vec{F} \cdot \Delta\vec{A}.$$

If S is a closed surface oriented outwards, we call this the flux out of $\vec{S}$.

How to compute? Parametrize surface with $\vec{r}(s,t)$. Divide up into small parallelograms. As in section 8.1, each has area $\|\vec{r}_s \times \vec{r}_t\| \Delta s \Delta t$. But direction of $\vec{r}_s \times \vec{r}_t$ is perpendicular to the parallelogram, so we can take $\Delta \vec{A} = \vec{r}_s \times \vec{r}_t \Delta s \Delta t$. Thus

$$\begin{aligned} \sum \vec{F} \cdot \Delta \vec{A} &\approx \sum \vec{F} \cdot (\vec{r}_s \times \vec{r}_t) \Delta s \Delta t \\ &\approx \int_a^b \int_c^d \vec{F} \cdot (\vec{r}_s \times \vec{r}_t)\, dt\, ds. \end{aligned}$$

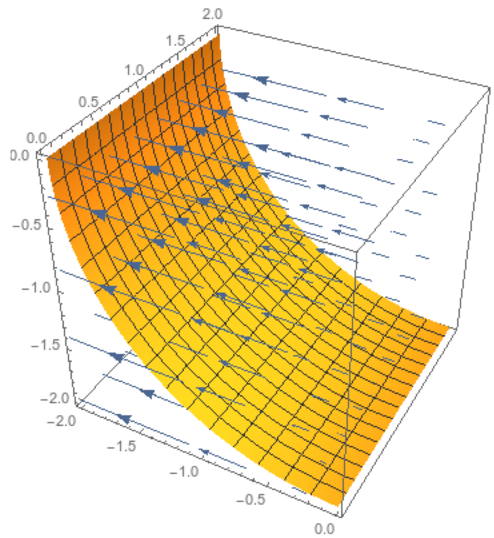
Example 8.10. Suppose we want to compute the flux of $\vec{F}(x,y,z) = x\vec{i}$ outwards through a portion of the cylinder of radius 2 centered on the z-axis with $x \leq 0$, $y \leq 0$, and $0 \leq z \leq 2$.



We can parametrize this by $\vec{r}(\theta, z) = (2\cos\theta, 2\sin\theta, z)$. From proposition 8.5 we know that $\vec{r}_\theta \times \vec{r}_z = (2\cos\theta, 2\sin\theta, 0)$ (and we check that this is oriented outwards), so we set up the integral

$$\begin{aligned} \int_0^2 \int_\pi^{3\pi/2} (2\cos\theta, 0, 0) \cdot (2\cos\theta, 2\sin\theta, 0)\, d\theta\, dz &= \int_0^2 \int_\pi^{3\pi/2} 4\cos^2\theta\, d\theta\, dz \\ &= \int_0^2 2\theta + \sin(2\theta)|_\pi^{3\pi/2}\, dz \\ &= \int_0^2 \pi\, dz = 2\pi. \end{aligned}$$

What if instead we take cylinder centered on y axis with $x \leq 0, z \leq 0, 0 \leq y \leq 2$?



We now parametrize it with $\vec{r}(\theta, y) = (2\cos\theta, y, 2\sin\theta)$ for $\pi \leq \theta \leq 3\pi/2$. We compute that

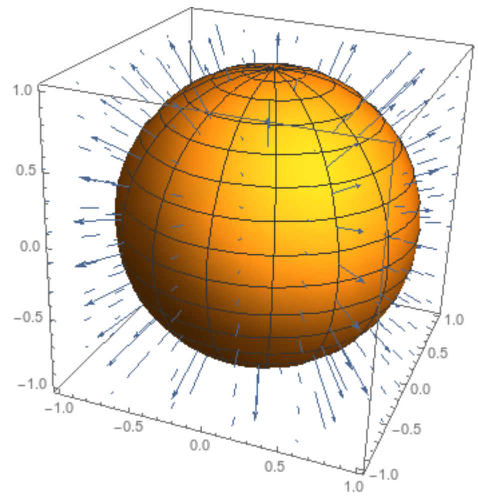
$$\vec{r}_\theta \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\theta & 0 & 2\cos\theta \\ 0 & 1 & 0 \end{vmatrix} = (-2\cos\theta, 0, -2\sin\theta).$$

But this vector is oriented inwards: if we take $\theta = \pi$ then our cross product vector is $(2, 0, 0)$ which points inwards from $\vec{r}(\pi, 0) = (-2, 0, 0)$. So we take the negative of this, and our cross product vector should be $\vec{r}_y \times \vec{r}_\theta = (2\cos\theta, 0, 2\sin\theta)$.

From there, we have a similar integral

$$\begin{aligned} \int_0^2 \int_\pi^{3\pi/2} (2\cos\theta, 0, 0) \cdot (2\cos\theta, 0, 2\sin\theta)\, d\theta\, dy &= \int_0^2 \int_\pi^{3\pi/2} 4\cos^2\theta\, d\theta\, dy \\ &= \int_0^2 2\theta + \sin(2\theta)|_\pi^{3\pi/2}\, dy \\ &= \int_0^2 \pi\, dy = 2\pi. \end{aligned}$$

Example 8.11 (Gauss's Law)**.** Flux of vector field $\frac{\vec{r}}{\|\vec{r}\|^3}$ through a sphere of radius R (oriented outwards).



We don't actually need to compute an integral here. The flux is always perpendicular to the surface of the sphere, so we have

$$\vec{F} \cdot d\vec{A} = \|\vec{F}\| \cdot \|d\vec{A}\| = \frac{dA}{\|\vec{r}^2\|} = \frac{dA}{R^2}$$

since we're evaluating on the sphere of radius R. Then the flux integral is just $\frac{1}{R^2} \int_S dA$ where the integral is just the area of the sphere of radius R, and thus the flux integral is equal to $4\pi R^2/R^2 = 4\pi$.

But suppose we want to compute the integral normally. We parametrize the unit sphere by $\vec{r}(\theta, \phi) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)$, and we worked out in proposition 8.5 that

$$\vec{r}_\phi \times \vec{r}_\theta = R^2\cos\theta\sin^2\phi\vec{i} + R^2\sin\theta\sin^2\phi\vec{j} + R^2\sin\phi\cos\phi\vec{k},$$

and this direction is oriented outwards. So the flux integral is

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{\pi} \frac{1}{R^3}(R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi) \cdot (R^2\cos\theta\sin^2\phi\vec{i} + R^2\sin\theta\sin^2\phi\vec{j} + R^2\sin\phi\cos\phi\vec{k})\,d\phi\,d\theta \\
&\quad = \int_0^{2\pi}\int_0^{\pi} \cos^2\theta\sin^3\phi + \sin^2\theta\sin^3\phi + \sin\phi\cos^2\phi\,d\phi\,d\theta \\
&\quad = \int_0^{2\pi}\int_0^{\pi} \sin\phi\,d\phi\,d\theta = \int_0^{2\pi} -\cos\phi|0^\pi\,d\theta = \int_0^{2\pi} 2\,d\theta = 4\pi.
\end{aligned}$$

Proposition 8.12. • *If S is the graph of $z = f(x, y)$ over R, then*

$$\int_S \vec{F} \cdot d\vec{A} = \int_R \vec{F}(x, y, f(x, y)) \cdot (-f_x\vec{i} - f_y\vec{j} + \vec{k})\,dx\,dy.$$

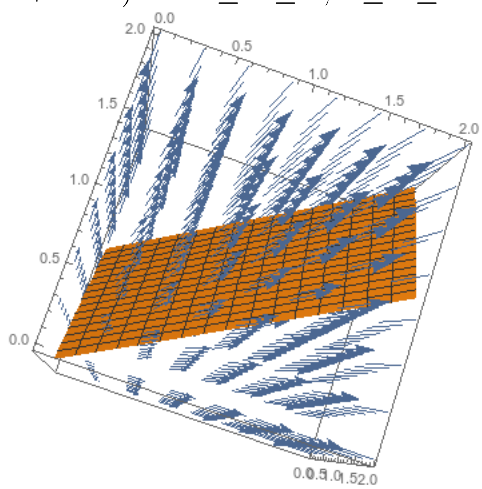
- *If S is a cylinder oriented away from the z-axis of radius R, then*

$$\int_S \vec{F} \cdot d\vec{A} = \int_T \vec{F}(R, \theta, z) \cdot (\cos\theta\vec{i} + \sin\theta\vec{j})R\,dz\,d\theta.$$

- *If S is a sphere of radius R oriented outwards, then*

$$\begin{aligned}\int_S \vec{F} \cdot d\vec{A} &= \int_S \vec{F} \cdot \frac{\vec{r}}{\|\vec{r}\|}\, dA \\ &= \int_T \vec{F}(R,\theta,\phi) \cdot (\sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k})R^2 \sin\phi\, d\phi\, d\theta.\end{aligned}$$

Example 8.13. Find flux of $\vec{F}(x,y,z) = x\vec{i} + y\vec{j}$ through the surface oriented downwards given by $\vec{r}(s,t) = (2s, s+t, 1+s-t)$ for $0 \le s \le 1, 0 \le t \le 1$.



We compute

$$\begin{aligned}\vec{r}_s(s,t) &= (2,1,1) \\ \vec{r}_t(s,t) &= (0,1,-1) \\ \vec{r}_s \times \vec{r}_t &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = (-1-1)\vec{i} + (0+2)\vec{j} + (2-0)\vec{k} = -2\vec{i} + 2\vec{j} + 2\vec{k}.\end{aligned}$$

But this vector is oriented upwards, so we take its opposite $2\vec{i} - 2\vec{j} - 2\vec{k}$. Then our integral is

$$\begin{aligned}\int_0^1 \int_0^1 (2s, s+t, 0) \cdot (2,-2,-2)\, ds\, dt &= \int_0^1 \int_0^1 2s - 2t\, ds\, dt \\ &= \int_0^1 s^2 - 2st|_0^1\, dt = \int_0^1 1 - 2t\, dt \\ &= t - t^2|_0^1 = 0.\end{aligned}$$

Thus there is no net flux.

8.3 Stokes's Theorem

First we have to define the boundary of a surface. I can give a technical definition: a point P is on the boundary of S if no open ball $B_\epsilon = \{Q : \|Q - P\| < \epsilon\}$ centered at P is entirely contained in S. But more generally we understand what the boundary of a surface is: it's the set of all the points on the edge.

We're going to want to worry about the orientation of a surface relative to its boundary. We want the orientations to be compatible: we determine compatibility via the right-hand rule. We pick an orientation for S by choosing a normal vector for every point. The boundary is oriented compatibly if the (clockwise or counterclockwise) circulation corresponding to this normal vector points in the same direction as the boundary.

Once our boundaries are oriented compatibly, we can make an argument very similar to Green's theorem. We can compute the circulation around the boundary with a line integral. Or, instead, we can compute the integral of the curl over the entire surface; this integrates all the circulation density, and thus gives us the total circulation.

Theorem 8.14 (Stokes). *If S is a smooth oriented surface with piecewise smooth oriented boundary C (with matching orientations of S and C), and $\vec{F}$ is a smooth vector field on an open region containing S and C, then*

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \nabla \times \vec{F} \cdot d\vec{A}.$$

Remark 8.15. If S is a region within the xy plane, then this is precisely Green's Theorem.

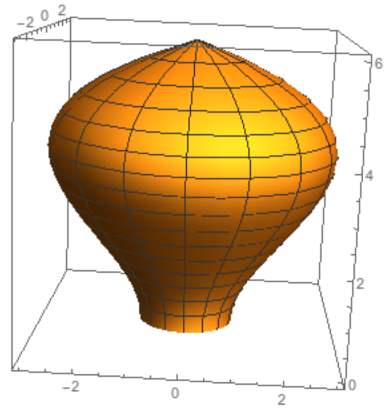
Example 8.16. Let $\vec{F}(x, y, z) = -2y\vec{i} + 2x\vec{j}$. Find $\int_C \vec{F} \cdot d\vec{r}$ where C is a circle parallel to yz plane centered in x axis.

Compute $\nabla \times F = 4\vec{k}$. Curl is parallel to circle, so flux of curl through disk is zero. Thus circulation is zero.

Where C is parallel to the xy plane, centered on z axis, radius r.

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \nabla \times F \cdot d\vec{A} = \|\nabla \times \vec{F}\| \cdot \text{area of } S = 4\pi r^2.$$

Example 8.17. Let's consider the surface of a lightbulb, whose base is given $x^2 + y^2 = 1$. Let $\vec{F}(x, y, z) = e^{z^2-2z}x\vec{i} + (\sin(xyz) + y + 1)\vec{j} + e^{z^2}\sin(z^2)\vec{k}$, and find flux of $\nabla \times \vec{F}$ outward through the lightbulb's surface.



Attempting to do this surface integral would clearly be terrible even if we had a good parametrization for the surface. Fortunately we can avoid this, since the entire boundary of the lightbulb is just $x^2 + y^2 = 1$. Thus

$$\begin{aligned}
\int_S \nabla \times \vec{F} \cdot d\vec{A} &= \int_C \vec{F} \cdot d\vec{r} \\
&= \int_0^{2\pi} \vec{F}(\cos\theta, \sin\theta, 0) \cdot (-\sin\theta, \cos\theta, 0)\, d\theta \\
&= \int_0^{2\pi} (\cos\theta, \sin\theta + 1, 0) \cdot (-\sin\theta, \cos\theta, 0)\, d\theta \\
&= \int_0^{2\pi} -\cos\theta\sin\theta + \cos\theta\sin\theta + \cos\theta\, d\theta \\
&= \int_0^{2\pi} \cos\theta\, d\theta = \sin\theta|_0^{2\pi} = 0.
\end{aligned}$$

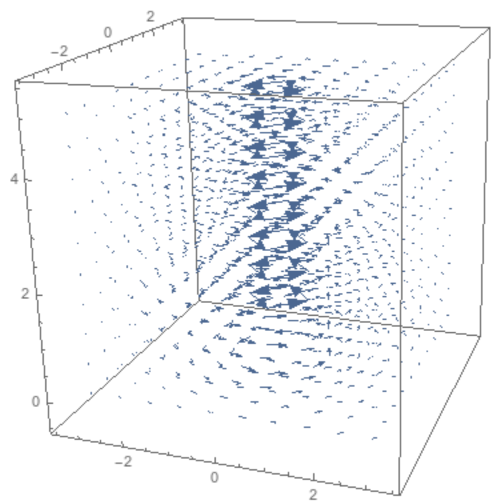
Notice that this doesn't depend on the specific shape of the lightbulb!

Stokes's theorem is particularly nice when we're studying an irrotational field—one with zero curl.

Example 8.18. Let

$$\vec{B}(x, y, z) = \frac{-y\vec{i} + x\vec{j}}{x^2 + y^2}.$$

This vector field is, among other things, the magnetic field induced by a current running down a wire along the z-axis.



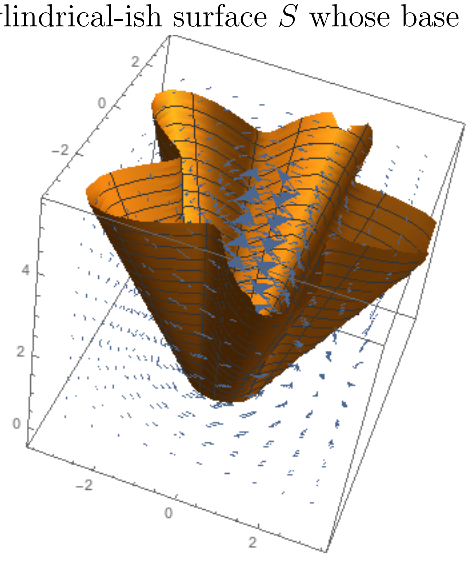
We saw this field in example 7.40, where we calculated that $\nabla \times \vec{B}(x, y, z) = \vec{0}$. Let's compute the circulation of $\vec{B}$ counterclockwise around C_2, a five-pointed star at the height $z = \pi + e$, centered at the z-axis.

We don't want to try to compute that directly; fortunately, we don't have to. We first might try using Stokes's theorem to integrate the curl (which is zero) over the interior; but we can't do that, because $\vec{B}$ isn't actually defined at $x = y = 0$. So we have to do something more complex.

Let's start by computing a relatively easy integral, over C_1, the counterclockwise circle of radius 1 in the xy plane. We parametrize this with $\vec{r}(t) = (\cos t, \sin t, 0)$, and we get the integral

$$\int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt = \int_0^{2\pi} dt = 2\pi.$$

Now let's consider the cylindrical-ish surface S whose base is C_1 and whose top is C_2. T



Then we see that the boundary of S (oriented outwards) is $C_1 - C_2$ (since we need to reverse the orientation of C_2 to match the orientation of S). Thus by Stokes's theorem we have

$$\int_S \nabla \times \vec{B} \cdot d\vec{A} = \int_{C_1 - C_2} \vec{B} \cdot d\vec{r}.$$

But $\nabla \times \vec{B} = \vec{0}$, so this tells us that

$$\begin{aligned} 0 &= \int_{C_1 - C_2} \vec{B} \cdot d\vec{r} \\ &= \int_{C_1} \vec{B} \cdot d\vec{r} - \int_{C_2} \vec{B} \cdot d\vec{r} \\ \int_{C_2} \vec{B} \cdot d\vec{r} &= \int_{C_1} \vec{B} \cdot d\vec{r} = 2\pi. \end{aligned}$$

Remark 8.19. This last example shows us something even more dramatic. The actual details of the curve C_2 are completely irrelevant; only the fact that it can combine with C_1 to form the boundary of a tube that doesn't intersect the z-axis. This argument shows that any closed curve that winds around the z-axis once will have an integral of $\pm 2\pi$, with the sign depending on the relative orientation of C_1 and C_2.

We can also turn this process backwards.

Definition 8.20. Let $\vec{G}$ be a vector field. If $\nabla \times \vec{F} = \vec{G}$, then we say that $\vec{F}$ is a *vector potential* for $\vec{G}$. If $\vec{G}$ has a vector potential, we say it is a *curl field*—which just means that it is the curl of some vector field.

Proposition 8.21. *If $\vec{G}$ is a curl field, then any two oriented surfaces with the same oriented boundary have the same flux integral.*

Proof. Suppose $\vec{G}$ is a curl field, and S_1 and S_2 are two oriented surfaces with the same oriented boundary C. Then by Stokes's theorem, we have

$$\int_{S_1} \vec{G} \cdot d\vec{A} = \int_C \vec{F}\, d\vec{r} = \int_{S_2} \vec{G} \cdot d\vec{A}.$$

□

This result should remind you of the path-independence result for line integrals. Two paths with the same boundary have the same integral if the vector field is a gradient field; two surfaces with the same boundary have the same integral if the vector field is a curl field.

We used the curl to test whether a vector field is a gradient field. Now we need a similar tool to test whether a vector field is a curl field.

9 Divergence and Differential Forms

Recall that in section 7 we defined the line integral, then looked for a way to avoid it. We saw that if a vector field $\vec{F}$ is "conservative" then line integrals are path independent—we can find some potential function f so that $\nabla f = \vec{F}$, and then evaluate f on the endpoints rather than integrating $\vec{F}$ over the whole curve. We then saw that if a vector field has $\nabla \times \vec{F} = 0$, it is conservative.

Now we want to do the same thing for surface integrals. In section 8 we defined the surface integral, then saw that if our field $\vec{G}$ is a curl field—that is, $\vec{G} = \nabla \times \vec{F}$ for some vector potential $\vec{F}$—then instead of computing the surface integral of $\vec{G}$ over a surface, we can integrate $\vec{F}$ over the boundary. But how can we tell when a field is a curl field?

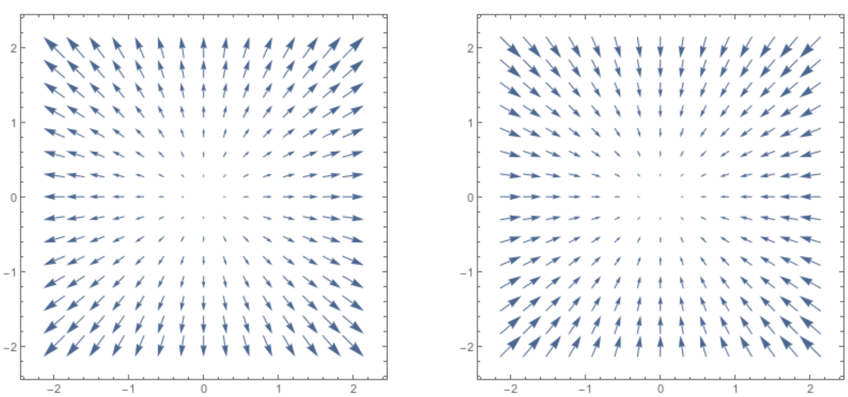
9.1 The divergence of a vector field

Just as we defined the curl to be the density of the circulation, which is the line integral of a vector field, we will define the *divergence* to measure the density of the flux. Thus divergence will measure the extent to which a vector field flows into or out of a region.

Definition 9.1. The *divergence* or *flux density* of a vector field is

$$\nabla \cdot \vec{F}(x, y, z) = \lim_{\text{volume}\to 0} \frac{\int_S \vec{F} \cdot d\vec{A}}{\text{volume of } S}$$

where S is a sphere centered at (x, y, z) oriented outwards.



The divergence at the origin on the left is positive; on the right it is negative.

Proposition 9.2. *We can compute the divergence with*

$$\begin{aligned}\nabla \cdot \vec{F} &= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right) \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.\end{aligned}$$

Proof. It doesn't actually matter whether we use a small sphere or a small box. So let's imagine computing the flux density of $\vec{F}$ over a small box oriented outwards, with side lengths $\Delta x, \Delta y, \Delta z$ and the corner closest to the origin at (x_0, y_0, z_0).

Then the box has six faces. Let's consider the top and bottom. The bottom face has vector $\Delta x \Delta y(-\vec{k})$. Since the box is small the vector field is approximately constant at the value $\vec{F}(x_0, y_0, z_0)$, so the integral over the bottom face is approximately

$$\vec{F}(x_0, y_0, z_0) \cdot (-\Delta x \Delta y)\vec{k} = -F_3(x_0, y_0, z_0)\Delta x \Delta y.$$

Looking at the top now, we see that the face has vector $\Delta x \Delta y \vec{k}$, but the vector field is now approximately $\vec{F}(x_0, y_0, z_0 + \Delta z)$. The integral is then approximately

$$\vec{F}(x_0, y_0, z_0 + \Delta z) \cdot \Delta x \Delta y \vec{k} = F_3(x_0, y_0, z_0 + \Delta z)\Delta x \Delta y.$$

Adding these two together gives

$$F_3(x_0, y_0, z_0 0 + \Delta z)\Delta x \Delta y - F_3(x_0, y_0, z_0)\Delta x \Delta y = \frac{F_3(x_0, y_0, z_0 + \Delta z) - F_3(x_0, y_0, z_0)}{\Delta z} \Delta x \Delta y \Delta z.$$

To compute the flux density, we divide by the volume, which is $\Delta x \Delta y \Delta z$. Then taking the limit gives us

$$\lim_{\Delta z \to 0} \frac{F_3(x_0, y_0, z_0 + \Delta z) - F_3(x_0, y_0, z_0)}{\Delta z} = \frac{\partial F_3}{\partial z}.$$

We run through the same calculations for the two faces on the side to get $\frac{\partial F_1}{\partial x}$ and the front and back to get $\frac{\partial F_2}{\partial y}$. Adding all three components together gives us

$$\nabla \cdot \vec{F}(x, y, z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

as desired.

□

Example 9.3. Let's compute the divergence of $\vec{F}(\vec{r}) = \vec{r}$ at the origin.

Even without doing any computations, we can see that the divergence must be positive, since the flux is definitely outwards.

First, let's use the geometric definition. A sphere of radius a has outward flux of $4\pi a^3$. We can see this by arguing that the vector field is always perpendicular to the sphere, so we have $\int_S \vec{F} \cdot d\vec{dA}$ is equal to the magnitude of $\vec{F}$ times the surface area of the sphere, which is $a \cdot 4\pi a^2 = 4\pi a^3$. Alternatively, we have

$$\text{Flux} = \int_S \vec{F} \cdot \frac{\vec{r}}{\|\vec{r}\|} \, dA = \int_S \|\vec{r}\| \, dA = a \int_S 1 \, da = a \cdot 4\pi a^2 = 4\pi a^3.$$

The volume is $4/3\pi a^3$, so the flux density is

$$\lim_{a\to 0} \frac{4\pi a^3}{4/3\pi a^3} = \lim_{a\to 0} 3 = 3.$$

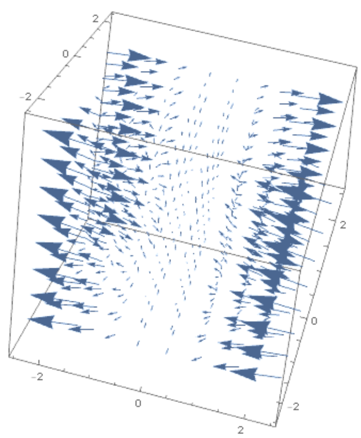
Alternately, we can compute with the algebraic definition. We have

$$\nabla \cdot \vec{F} = \nabla \cdot (x\vec{i} + y\vec{j} + z\vec{k}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3.$$

As usual, you can see that we'd much prefer to compute the divergence using the algebraic definition, rather than the geometric definition.

Example 9.4. Let $\vec{F}(x, y, z) = x^2y\vec{i} + \cos(z)\vec{j} + \sin(z)\vec{k}$. Then

$$\nabla \cdot \vec{F}(x, y, z) = 2xy + 0 + \cos(z).$$



Let $\vec{G}(x, y) = -y\vec{i} + x\vec{j}$. Then $\nabla \cdot \vec{G} = 0 + 0 = 0$. $\vec{G}$ is rotating, but it has zero divergence since there's no net flux into or out of any region.

Any constant vector field has zero divergence since the exact same amount of fluid is entering and leaving every point.

Definition 9.5. We say that $\vec{F}$ is *divergence free* or *solenoidal* or *incompressible* if $\nabla \cdot \vec{F} = 0$ whenever $\vec{F}$ is defined.

Physically, this means that the density of fluid is conserved—on net it isn't flowing into or out of any region.

Example 9.6. Let $\vec{E} = \frac{\vec{r}}{\|\vec{r}\|^p}$. Let's find the divergence, and determine for what p this field is solenoidal.

We have

$$\begin{aligned}
\frac{\partial E_1}{\partial x} &= \frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{p/2}} \\
&= \frac{(x^2+y^2+z^2)^{p/2} - x\frac{p}{2}(x^2+y^2+z^2)^{p/2-1}(2x)}{(x^2+y^2+z^2)^p} \\
&= \frac{(x^2+y^2+z^2)^{p/2} - px^2(x^2+y^2+z^2)^{p/2-1}}{(x^2+y^2+z^2)^p} \\
&= \frac{(x^2+y^2+z^2) - px^2}{(x^2+y^2+z^2)^{p/2+1}} \\
\frac{\partial E_2}{\partial y} &= \frac{(x^2+y^2+z^2) - py^2}{(x^2+y^2+z^2)^{p/2+1}} \\
\frac{\partial E_3}{\partial z} &= \frac{(x^2+y^2+z^2) - pz^2}{(x^2+y^2+z^2)^{p/2+1}} \\
\nabla \cdot \vec{E} &= \frac{3(x^2+y^2+z^2) - px^2 - py^2 - pz^2}{(x^2+y^2+z^2)^{p/2+1}} \\
&= \frac{3-p}{(x^2+y^2+z^2)^{p/2}} = \frac{3-p}{\|\vec{r}\|^p}.
\end{aligned}$$

Thus $\vec{E}$ is solenoidal if and only if $p = 3$. (Recall that in this case, we have an inverse-square law, as appears in equations for electromagnetism. In fact, all magnetic fields are solenoidal).

Proposition 9.7. *If $\vec{G} = \nabla \times \vec{F}$, then $\nabla \cdot \vec{G} = 0$.*

If $\vec{G}$ be a vector field defined everywhere and $\nabla \cdot \vec{G} = 0$, then $\vec{G}$ is a curl field, that is, there exists a vector field $\vec{F}$ such that $\nabla \times \vec{F} = \vec{G}$.

Proof. Suppose there is a vector field $\vec{F}$ such that $\nabla \times \vec{F} = \vec{G}$. Then at any point, we know that

$$\nabla \cdot \vec{G} = \lim \frac{\int_S \vec{G} \cdot d\vec{A}}{\text{Volume}}.$$

But by Stokes's theorem, we know that

$$\int_S \vec{G} \cdot d\vec{A} = \int_C \vec{F} \cdot d\vec{r}$$

where C is the boundary of S. But S is a sphere, so the boundary is empty, and this integral must be zero.

(You can also draw an arbitrary closed curve along S, and compute the flux integrals of the two pieces S is divided into; they must be equal except for sign, again by Stokes's theorem).

Then

$$\nabla \cdot \vec{G} = \lim \frac{0}{\text{Volume}} = \lim 0 = 0.$$

Conversely, suppose that $\nabla \cdot \vec{G} = 0$. Then we define a function

$$\vec{F}(\vec{r}) = \int_0^1 \vec{G}(t\vec{r}) \times t\vec{r}\, dt.$$

This function is defined everywhere, and after some annoying algebra (and the knowledge that the curl operator commutes with integrals) we can check that $\nabla \times \vec{F} = \vec{G}$. Notice that here we need $\vec{G}$ to be defined everywhere, in order for this integral to be consistently defined. □

Remark 9.8. Compare both this proposition and its proof with proposition 7.36 in section 7.4.

Example 9.9. We saw earlier that $\vec{G}(x, y) = -y\vec{i} + x\vec{j}$ is solenoidal. Let's find its vector potential.

We know that $\nabla \times \vec{F} = \vec{G}$. In particular, this means that

$$\begin{aligned} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} &= -y \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} &= x \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= 0. \end{aligned}$$

It's hard to figure out where to start with this, because we have a lot of information to work with.

In fact we have a lot of degrees of freedom, since if f is any scalar function then $\nabla \times (\nabla f) = \vec{0}$ and thus $\nabla \times (\vec{F} + \nabla f) = \vec{G}$, so there are infinitely many options. By choosing a f so that $\frac{\partial f}{\partial z} = -F_3$ we can assume that $F_3 = 0$, and our equations reduce to

$$\begin{aligned} -\frac{\partial F_2}{\partial z} &= -y \\ \frac{\partial F_1}{\partial z} &= x \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= 0. \end{aligned}$$

Then we have $F_2(x, y, z) = yz + g(x, y)$, and $F_1(x, y, z) = xz + h(x, y)$. Then the third equation tells us that $\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y}$, and so we can take $\vec{F}(x, y, z) = xz\vec{i} + yz\vec{j}$.

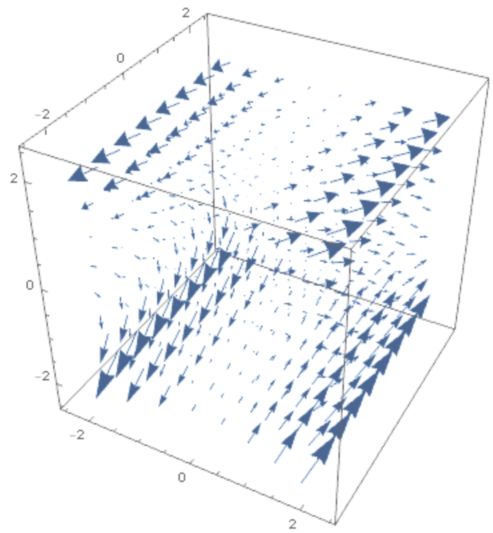
We can check that

$$\nabla \times (xz\vec{i} + yz\vec{j}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz & 0 \end{vmatrix} = -y\vec{i} + x\vec{j} + (0-0)\vec{k} = \vec{G}(x, y, z).$$

Example 9.10. Let $\vec{G}(x, y, z) = (x^2, 3xz^2, -2xz)$. Then $\nabla \cdot \vec{G} = 2x + 0 - 2x = 0$, so $\vec{G}$ is solenoidal. We can work out that $\vec{F} = \nabla \times (xz^3, -x^2z, 0)$ by computing

$$\begin{aligned} -\frac{\partial F_2}{\partial z} &= x^2 \\ \frac{\partial F_1}{\partial z} &= 3xz^2 \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= -2xz. \end{aligned}$$

Then $F_2(x, y, z) = -x^2z + f(x, y)$ and $F_1(x, y, z) = xz^3 + g(x, y)$. The third equation gives us that $-2xz + \frac{\partial f}{\partial x} - \frac{\partial g}{\partial y} = -2xz$ so we can take $f = g = 0$.



9.2 The Divergence Theorem

Now we're ready to state the higher-dimension analogue of Green's Theorem.

Theorem 9.11 (Divergence Theorem). *Let W be a solid three-dimensional region whose boundary S is a piecewise smooth surface oriented outwards, and $\vec{F}$ a smooth vector field on an open region containing S and W. Then*

$$\int_S \vec{F} \cdot d\vec{A} = \int_W \nabla \cdot \vec{F} \, dV.$$

Proof. This proof is basically the same as the proof of Green's theorem.

The left-hand integral is the flux out of the entire boundary. We can approximate the right-hand integral by dividing the region up into small cubes; the divergence in each cube is approximately the average flux out of that cube. Taking the integral adds up the flux from each cube, and we get the total flux out of W. □

Example 9.12. Let $W = \{(x, y, z) : -1 \leq x, y, z \leq 1\}$ be a cube with side length 2 centered at the origin, and let S be its boundary. Set $\vec{F}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$. What is the flux of $\vec{F}$ out of S?

Computing the flux integral directly would involve parametrizing six separate surfaces. Instead we can compute $\nabla \cdot \vec{F} = 1 + 1 + 1 = 3$, so

$$\int_S \vec{F} \cdot d\vec{A} = \int_W \nabla \cdot \vec{F}\, dV = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} 3\, dx\, dy\, dz = 24.$$

Suppose instead the vector field is $\vec{F}(x, y, z) = xy\vec{i} + yz\vec{j} + xyz\vec{k}$. Then we compute $\nabla \vec{F}(x, y, z) = y + z + xy$, and our total flux is

$$\begin{aligned} \int_S \vec{F} \cdot d\vec{A} &= \int_W \nabla \cdot \vec{F}\, dV \\ &= \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} y + z + xy\, dx\, dy\, dz = 0. \end{aligned}$$

Example 9.13. Compute $\int_S \vec{F} \cdot d\vec{A}$ where $\vec{F}(x, y, z) = (x^2yz + y^2z)\vec{i} + xy^2z\vec{j} + (x^2 + y^2)\vec{k}$, and S is the surface of the portion of the unit sphere in the octant $x, y, z \geq 0$. Notice this surface has four pieces!

We could parametrize each piece and compute the surface integral over it, but that seems difficult. Instead we compute

$$\begin{aligned} \nabla \cdot \vec{F}(x, y, z) &= 2xyz + 2xyz + 0 = 4xyz \\ \int_S \vec{F} \cdot d\vec{A} &= \int_W 4xyz \cdot dV \\ &= \int_0^1 \int_0^{2\pi} \int_0^{\pi} 4(\rho \cos\theta \sin\phi)(\rho \sin\theta \sin\phi)(\rho \cos\phi)\rho^2 \sin\phi\, d\phi\, d\theta\, d\rho \\ &= \int_0^1 \int_0^{2\pi} \int_0^{\pi} 4\rho^4 \cos\theta \sin\theta \sin^3\phi \cos\phi\, d\phi\, d\theta\, d\rho \\ &= \int_0^1 \int_0^{2\pi} \rho^4 \cos\theta \sin\theta \sin^4\phi|_0^{\pi}\, d\theta\, d\rho = 0. \end{aligned}$$

Proposition 9.14. *If $\vec{F}$ is a solenoidal vector field defined on W, and S is the boundary of W, then $\int_S \vec{F} \cdot d\vec{A} = 0$.*

Proof.

$$\int_S \vec{F} \cdot d\vec{A} = \int_W \nabla \cdot \vec{F}\, dV = \int_W 0\, dV = 0.$$

□

Example 9.15. Let $\vec{F}(\vec{r}) = \frac{\vec{r}}{\|\vec{r}\|^3}$. Let's compute the surface integral of $\vec{F}$ over S, the ellipsoid $x^2 + 4y^2 + 9z^2 = 25$ oriented outwards.

We don't want to parametrize this. But it's easy to compute the flux over the sphere oriented outwards. We see that $\vec{F}$ is always perpendicular to the sphere, so the surface integral is $\|\vec{F}\| \cdot 4\pi = 4\pi$.

Now consider the region of space W bounded on the inside by a sphere of radius 1, and bounded on the outside by S. Then by the divergence theorem, we have

$$\int_W \nabla \cdot F\, dV = \int_{S-T} \vec{F} \cdot d\vec{A}$$

where T is the unit sphere oriented outwards; we have the integral over $S - T$ because we want the boundary oriented away from the region, and thus we take the sphere oriented inwards.

But we saw in example 9.6 that $\nabla \cdot \vec{F}(\vec{R}) = 0$. Thus we have

$$\begin{aligned} 0 &= \int_{S-T} \vec{F} \cdot d\vec{A} \\ &= \int_S \vec{F} \cdot d\vec{A} - \int_T \vec{F} \cdot d\vec{A} \\ \int_S \vec{F} \cdot d\vec{A} &= \int_T \vec{F} \cdot d\vec{A} = 4\pi. \end{aligned}$$

Thus the flux integral through S is equal to 4π.

9.3 The Three Theorems

Over this course, we have defined three different derivative operators:

- The gradient ∇ takes in a scalar function and outputs a vector field.
- The curl $\nabla\times$ takes in a vector field and outputs another vector field.
- The divergence $\nabla\cdot$ takes in a vector field and outpus a scalar function.

We also discovered the following relationships among the derivative operators:

1. $\nabla \times \nabla f = \vec{0}$

2. If $\vec{F}$ is defined everywhere and $\nabla \times \vec{F} = \vec{0}$, then there is a f with $\nabla f = \vec{F}$.

3. $\nabla \cdot \nabla \times G = \vec{0}$

4. If $\vec{G}$ is defined everywhere and $\nabla \cdot \vec{G} = 0$ then there is a $\vec{F}$ with $\nabla \times \vec{F} = \vec{G}$.

Notice that the first two statements look just like the last two statements. Whenever we have a collection of theorems like this that all look similar, we should ask what they're a specialcase of.

We also developed three major theorems that let us convert one integral into another.

$$\int_C \nabla f \cdot d\vec{r} = f(Q) - f(P)$$
$$\int_S \nabla \times \vec{F} \cdot d\vec{A} = \int_C \vec{F} \cdot d\vec{r}$$
$$\int_W \nabla \cdot \vec{F}\, dV = \int_S \vec{F} \cdot d\vec{A}$$

Again, these theorems all look similar: they each tell us that the integral over a region of the derivative of a function, is equal to the integral over the boundary of the original function. Again, we want to figure out the right statement to generalize all these theorems. The correct statement requires us to understand differential forms.

9.4 Differential Forms

differential equation (indefinite integral); Lebesgue integral (unsigned definite integral); integration of forms (signed definite integral).

How does an integral work? Let's think about the single-variable case. If we're doing a path integral over a path r, we compute something like

$$\int_r \vec{F} \cdot d\vec{r} = \lim \sum \vec{F}(\vec{x}_i) \cdot \vec{r}\,'(\vec{x}_i) \Delta x.$$

What we're doing here is computing the infinitesimal work over a very small straight-line movement, and then adding up all of these infinitesimals. In order to do this, we need two things: we need a path to integrate over, and we need some way of computing the work done over a small section of that path. We want to generalize both of those things.

Definition 9.16. A k-dimensional *parametrized oriented manifold* in an ambient space $\mathbb{R}^n$ is a function $\vec{r} : [0,1]^k \to \mathbb{R}^n$. An *oriented manifold* is any surface that can be broken into pieces, each of which can be parametrized in this way.

Remark 9.17. In practice we don't always choose parametrizations where all of our bounds go from 0 to 1. But we could if we wanted to, and the theory is easier to work with if we make that assumption. It doesn't change anything important.

In the one-dimensional case, we divided our curve up into infinitesimal line segments with lengths Δx. In the k-dimensional case we want to chop things up into infinitesimal squares or cubes or hypercubes. We will represent an infinitesimal square with $\Delta x_1 \wedge \Delta x_2$, a cube with $\Delta x_1 \wedge \Delta x_2 \wedge \Delta x_3$, and so on. (The wedge represents an "exterior product", but don't worry too much about what that means).

In our path integral, we needed some function that would take in an infinitesimal path, at a location in space, and tell us how much work was done by moving over that infinitesimal path. Thus, we needed a function that takes in a point $\vec{x}$ and an infinitesimal vector $\Delta\vec{x}$, and outputs an amount of work. Thus we need a function $\omega_{\vec{x}} : \mathbb{R}^n \to \mathbb{R}$.

Further, we want this function to be *linear*, which means that it commutes with addition and multiplication:

1. $\omega_{\vec{x}}(\Delta\vec{x}_1 + \Delta\vec{x}_2) = \omega_{\vec{x}}(\Delta\vec{x}_1) + \omega_p(\Delta\vec{x}_2)$
2. $\omega_{\vec{x}}(r\Delta\vec{x}) = r\omega_{\vec{x}}(\Delta\vec{x})$.

So where can we get these functions from? It turns out that every function with this property can be given by a dot product, and there is some $\vec{F}$ such that $\omega_{\vec{x}}(\vec{v}) = \vec{F}(\vec{x}) \cdot \vec{v}$. Thus we have the usual path integral formulation:

$$\int_r \omega \approx \sum \omega_{\vec{x}_i}(\Delta\vec{x}_i) = \sum \vec{F}(\vec{x}_i) \cdot \Delta\vec{x}_i \approx \sum \vec{F}(\vec{x}_i) \cdot \vec{r}'(\vec{x}_i)\Delta x_i \approx \int_r F \cdot d\vec{r}.$$

We'd like to generalize this idea, to the idea of a *differential form.*

Definition 9.18. A k-form at a point $\vec{x}$ is a multilinear function $\omega_{\vec{x}} : (\mathbb{R}^n)^k$ to $\mathbb{R}$. That is, a k form takes in k vectors and outputs a real number, and satisfies the axioms

1. $\omega_{\vec{x}}(\Delta\vec{x}_1 \wedge \cdots \wedge \Delta\vec{x}_k) + \omega_{\vec{x}}(\Delta\vec{y}_1 \wedge \cdots \wedge \Delta\vec{x}_k) = \omega_{\vec{x}}((\Delta\vec{x}_1 + \Delta\vec{y}_1) \wedge \cdots \wedge \Delta\vec{x}_k)$
2. $r\omega_{\vec{x}}(\Delta\vec{x}_1 \wedge \cdots \wedge \Delta\vec{x}_k) = \omega_{\vec{x}}(r\Delta\vec{x}_1 \wedge \cdots \wedge r\Delta\vec{x}_k)$

We also require that $\omega_{\vec{x}}\Delta\vec{x} \wedge \Delta\vec{x} = 0$ for any $\vec{x}$. This represents the idea that a "parallelogram" whose two edges are given by the same vector has zero area.

Remark 9.19. We can also define a zero-form, which at each point takes in no vectors at all and outputs a real number. A zero-form is then just a scalar function.

We can get one more fact out of this wedge product: we have

$$\begin{aligned} 0 &= (\Delta\vec{x}_1 + \Delta\vec{x}_2) \wedge (\Delta\vec{x}_1 + \Delta\vec{x}_2) \\ &= \Delta\vec{x}_1 \wedge \Delta\vec{x}_1 + \Delta\vec{x}_1 \wedge \Delta\vec{x}_2 + \Delta\vec{x}_2 \wedge \Delta\vec{x}_1 + \Delta\vec{x}_2 \wedge \Delta\vec{x}_2 \\ &= \Delta\vec{x}_1 \wedge \Delta\vec{x}_2 + \Delta\vec{x}_2 \wedge \Delta\vec{x}_1 \\ \Delta\vec{x}_1 \wedge \Delta\vec{x}_2 &= -\Delta\vec{x}_2 \wedge \Delta\vec{x}_1. \end{aligned}$$

Thus the wedge product is anticommutative.

Every differential k-form can be written $f(\vec{x})dx_{a_1} \wedge \cdots \wedge dx_{a_k}$ for some scalar function $f(\vec{x})$ and some $a_i \in \{1, \dots, n\}$. (In algebraic language, the set $\{dx_{a_1} \wedge \cdots \wedge dx_{a_k}\}$ forms a basis for the space of differential k-forms).

In $\mathbb{R}^3$, a 1-form is $fdx + gdy + hdz$; we have actually seen this notation in section 8.1 when we talked about differential notation. Similarly, a 2-form is $fdxdy + gdxdz + hdydz$. Every 3-form is $fdxdydx$.

Definition 9.20. We define the *integral* of a k-form ω over a k-dimensional oriented manifold r by

$$\int_r \omega = \sum \omega_{\vec{x}_i}(\Delta\vec{x}_{i,1} \wedge \dots \Delta\vec{x}_{i,k}).$$

We've introduced a bunch of new notation now. What does that get us? Well, first, we've figured out how to extend our definitions to integrals in more than three dimensions, at least sort of. But it actually gets us a lot more after we define the derivative.

If f is a scalar function, then the derivative df at the point $\vec{x}$ is a linear function such that $f(\vec{x} + \vec{v}) \approx f(\vec{x}) + df_{\vec{x}}(\vec{v})$. (Algebraically this is given by the gradient). But this is a linear function that takes in a vector and outputs a scalar, and thus is a 1-form. We can generalize this:

Definition 9.21. Let $\omega = fdx_{a_1} \wedge \cdots \wedge dx_{a_k}$ be a k-form. Then we define the derivative of ω to be the $k+1$-form

$$d\omega = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \wedge dx_{a_1} \wedge \cdots \wedge dx_{a_k}.$$

Proposition 9.22. *Let ω be a k-form and α a ℓ-form. Then*

1. $d(d\omega) = 0$
2. $d(\omega \wedge \alpha) = d\omega \wedge \alpha + (-1)^k(\omega \wedge d\alpha)$.

We now can explain all of the patterns we saw in section 9.3. We have already seen that 0-forms are scalars, and we can identify 1-forms with vector fields. When the ambient space is $\mathbb{R}^3$, we can also identify 2-forms with vector fields, and 3-forms with scalar functions. (For instance, a 3-form in $\mathbb{R}^3$ is always given by $f dx \wedge dy \wedge dz$, so we can identify it purely by the function f).

We had three derivatives: the gradient, the curl, and the divergence. In $\mathbb{R}^3$, then the derivative of a 0-form is given by the gradient; the derivative of a 1-form is given by the curl; and the derivative of a 2-form is given by the divergence.

We had the patterns given by the Fundamental Theorem of Line Integrals, Stokes's theorem, and the Divergence Theorem. These are all generalized into the generalized Stokes's theorem.

Theorem 9.23 (Stokes)**.** *If M is an oriented $k+1$-dimensional manifold with boundary ∂M an oriented k-dimensional manifold, and ω is a k-form, then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

This makes it important to ask when a $(k+1)$-form is the derivative of some k-form.

Definition 9.24. We say a k-form ω is *closed* if $d\omega = 0$. A closed 1-form is irrotational and a closed 2-form is solenoidal.

We say that ω is *exact* if there is a $(k-1)$-form α with $d\alpha = \omega$. An exact 1-form is conservative and an exact 2-form is a curl field.

We can see that every exact form must also be closed, since then $d\omega = d(d\alpha) = 0$. It is not the case, that we have seen in class, that every closed form is exact; however, a closed form defined everywhere on Euclidean space must be exact.

More generally, every closed form on a domain with no holes is exact; but a domain with holes in it will have closed forms that are not exact. This means we can use the difference between closed forms and exact forms to measure the extent to which our domain has holes.

Definition 9.25. If α, β are closed k-forms on a manifold M, then we say that α and β are *cohomologous* if $\alpha - \beta$ is an exact form. This is an equivalence relation on closed k-forms.

We define the kth *de Rham cohomology group* of M, written $H^k_{dR}(M)$, to be the set of equivalence classes of closed k-forms under the cohomology relation. In fact this set forms a *group* under the operation of addition.

If $H^k_{dR}(M)$ contains exactly one element, then all closed forms are exact. This implies that every k-dimensional sphere can be contracted through M to become a single point.

$H^1_{dR}(M)$ contains exactly one element, we say that M is *simply connected.*

This doesn't seem that useful, but the theory of algebraic topology and the Mayer-Vietoris sequence allows us to compute these cohomology groups in a relatively easy and calculus-free way. Unfortunately, We won't be discussing that here.

Finally, we should mention the pull-back. Suppose M and N are manifolds and $\phi : M \to N$ is a function between them. If we have a k-form ω defined on N, then we can use it to define a form $\phi^*(\omega)$ on M that satisfies

$$\int_{\phi(M)} \omega = \int_M \phi^*(\omega).$$

We call this form the *pull-back* of ω along ϕ, and define it essentially by plugging points of M into ϕ before plugging that into ω; thus if f is a 0-form, we have $\phi^* f(\vec{x}) = f(\phi(\vec{x}))$; and if ω is a 1-form we have

$$(\phi^*\omega)_{\vec{x}}(\vec{v}) = \omega_{\phi(\vec{x})}(\phi(\vec{v})).$$

The pull-back conveniently satisfies the relationships $\phi^*(\omega \wedge \alpha) = (\phi^*\omega) \wedge (\phi^*\alpha)$ and $d(\phi^*\omega) = \phi^*(d\omega)$; and from the pull-back, we can recover the change-of-variable integral formulas we use for u-substitution and in section 6.3.