1 Zero-Order Approximation: Continuity and Limits

Let's start with an easy question:

Question 1.1. What is the square root of four?

Everyone can probably tell me that the answer is "two". So now let's do a harder one:

Question 1.2. What is the square root of five?

Without a calculator, you probably can't tell me the answer. But you should be able to make a pretty good guess. Five close to four; so $\sqrt{5}$ should be close to two.

We call this sort of estimate a *zeroth-order approximation*. In a zeroth-order approximation, we only get to use one piece of information: the value of our function at a specific number. Then we use that information to estimate its value at nearby numbers.

We can only do so good a job with that limited amount of information, but we can still do a surprising amount.

1.1 Continuous Functions

Example 1.3. Suppose $f(1) = 36, f(2) = 35, f(3) = 38, f(4) = 38$. What can we say to estimate $f(5)$?

From looking at the data we have, it seems like $f(5)$ should be 38 or 39, probably. But it's actually 45. These are the low temperatures in Pasadena for the first five days of this year.

Often tomorrow's temperature will be similar to today's temperature. But there's no guarantee.

This example shows that we can't always do what we did with $\sqrt{5}$. Some functions jump around too much for this sort of approximation thing to work; values of similar inputs don't have similar outputs.

We don't like these functions, precisely because they're hard to think about or understand. So we're mostly going to look at functions that we *can* approximate effectively.

Definition 1.4 (Informal)**.** We say a function f is *continuous* at a number a if whenever x is close to a, then $f(x)$ is close to $f(a)$.

In other words, for a continuous function, when x and a are close together, then $f(a)$ is a decent approximation for $f(x)$.

Another way to think of this is that the function f is continuous at a if it doesn't "jump" at a.

There are a few different ways for a function to not be continuous at a given number. I will categorize these more carefully in a couple days, but right now I want to show you a few different things that can happen.

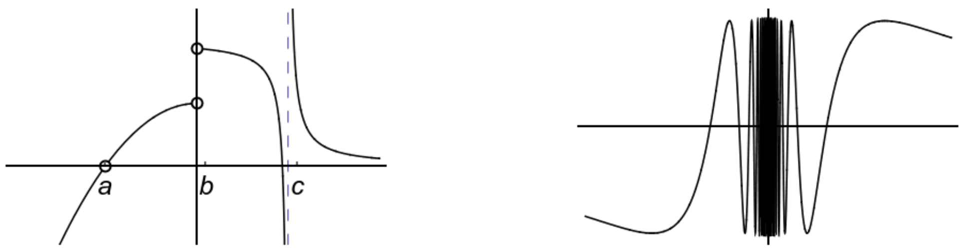


Figure 1.1: Left: a: removable discontinuity; b: jump discontinuity; c: infinite discontinuity. Right: bad discontinuity

Some functions get even worse than that. My two favorite discontinuous functions are:

$$\chi(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases} \qquad T(x) = \begin{cases} 1/q & x = p/q \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

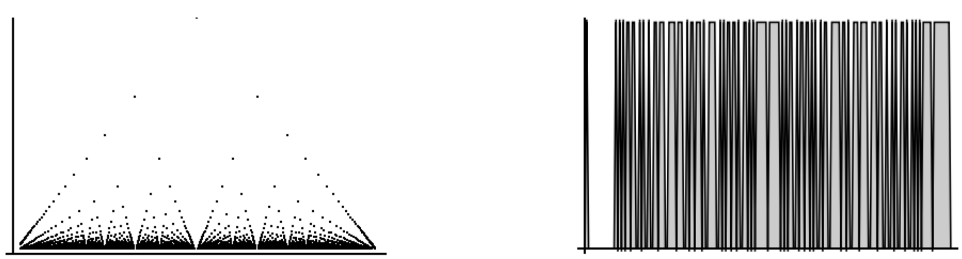


Figure 1.2: Left: $\chi(x)$ is really discontinuous. Right: $T(x)$ is really really discontinuous

In fact, in some sense "most functions" aren't at all continuous. If you found away to choose $f(x)$ completely at random for each real number x, you would get a spectacularly discontinuous function. But you would never actually be able to describe it sensibly.

But for the most part this isn't a problem. Most of the functions that we can easily describe are continuous most of the time. And so when approximating functions we don't understand, we often assume it's reasonably continuous.

Fact 1.5. *Any reasonable function given by a reasonable single formula is continuous at any number for which it is defined.*

In particular, any function composed of algebraic operations, polynomials, exponents, and trigonometric functions is continuous at every number in its domain.

If a function is continuous at every number in its domain, we just say that it is continuous. Note, importantly, that a continuous function doesn't have to be continuous at every real number.

Example 1.6. The function

$$f(x) = \frac{x^3 - 5x + 1}{(x-1)(x-2)(x-3)}$$

is "reasonable", so it is continuous. This means that it is continuous exactly on its domain, which is $\{x : x \neq 1, 2, 3\}$.

Example 1.7. Where is $\sqrt{1 + x^3}$ continuous?

Answer: Root functions are continuous on their domains. $1 + x^3 \geq 0$ when $x \geq -1$ so the function is continuous on its domain, $[-1, +\infty)$.

Remark 1.8. Sometimes we might also talk about functions that are "continuous from the right" at a. This means that $f(a)$ is a good approximation of $f(x)$ if x is close to a and also bigger than—and thus to the right of—a.

1.2 Limits

In order to understand continuity better, it's helpful to turn the question around and look at things from the opposite direction. (This is a trick that's often useful in math). So instead of asking whether we can estimate $f(x)$ given $f(a)$, we'll turn this around. If we know $f(x)$ for every x near a, what can we say about $f(a)$?

Definition 1.9. Suppose a is a real number, and f is a function which is defined for all x "near" the number a. We say "The *limit* of $f(x)$ as x approaches a is L," and we write

$$\lim_{x \to a} f(x) = L,$$

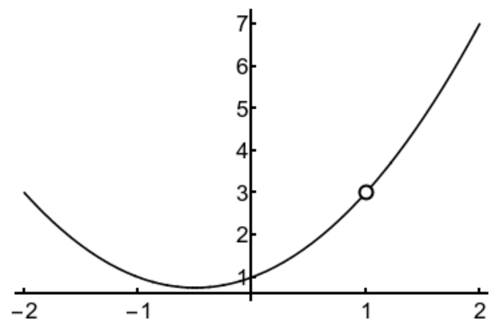
if we can make $f(x)$ get as close as we want to L by picking x that are very close to a.

Graphically, this means that if the x coordinate is near a then the y coordinate is near L. Pictorially, if you draw a small enough circle around the point $(a, 0)$ on the x-axis and look at the points of the graph above and below it, you can force all those points to be close to L.

Notice that we're trying to use knowing $f(x)$ to tell us what happens near a. So we specifically ignore the value of $f(a)$ even if we already know it.

Example 1.10. Let's consider the function $f(x) = \frac{x^3-1}{x-1}$. We can see the graph below. Notice that the function isn't defined at $a = 1$, so $f(1)$ is meaningless and we can't compute it.

But f is defined for all x near 1, so we can compute the limit. Looking at the graph and estimating suggests that when x gets close to 1, then $f(x)$ gets close to 3, and so we can say that $\lim_{x\to 1} f(x) = 3$.



That last example worked, but we basically just eyeballed it. We want a way to actually justify our claims. We can do that using two core principles. The first is what I call the Almost Identical Functions property.

Lemma 1.11 (Almost Identical Functions)**.** *If $f(x) = g(x)$ on some open interval $(a-d, a+d)$ surrounding a, except possibly at a, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ whenever one limit exists.*

This tells us that two functions have the same limit at a if they have the same values near a. This makes sense, because the limit only depends on the values near a.

How does this help us? Ideally, we take a complicated function and replace it with a simpler function.

Example 1.12. Above, we looked at the function $f(x) = \frac{x^3-1}{x-1}$. You may know that we can factor the numerator; thus we in fact have $f(x) = \frac{(x-1)(x^2+x+1)}{x-1}$.

At this point you probably want to cancel the $x-1$ term on the top and the bottom. But in fact that would change the function! For $f(1)$ isn't defined. But the function $g(x) = x^2+x+1$ is perfectly well-defined at $a = 1$. Thus $f(1) \neq g(1)$, and so f and g can't be the same function.

However, they do give the same value if we plug in any number other than 1. If $y \neq 1$ then $y - 1 \neq 0$, so we have

$$f(y) = \frac{(y-1)(y^2+y+1)}{y-1} = y^2 + y + 1 = g(y).$$

Thus f and g aren't the same, but they are *almost* the same. So lemma 1.54 tells us that $\lim_{x\to 1} f(x) = \lim_{x\to 1} g(x)$.

However, this doesn't actually do everything we want it to do. We've replaced a complicated function $f(x) = \frac{x^3-1}{x-1}$ with a simpler function $g(x) = x^2 + x + 1$, but we still haven't figured out what to do with that function.

This leads to our second principle. We started off talking about continuous functions, and said that if f is continuous at a, then $f(a)$ is a good estimate for $f(x)$ when x is near to a. In other words, when x is near a then $f(x)$ is near $f(a)$—so $\lim_{x\to a} f(x) = f(a)$.

Definition 1.13 (Formal)**.** We say that f is continuous at a if $\lim_{x\to a} f(x) = f(a)$.

This definition works in both directions. If we want to know whether a function is continuous, we can check its limits; and if we want to know the limit of a continuous function, we can find it by plugging in.

This really is the same as the less formal definition we gave in section 1.1. There, we said that f is continuous if $f(a)$ is a good approximation for $f(x)$; here we say that f is continuous if $f(x)$ is a good approximation for $f(a)$. This also clarifies *how good* the approximation needs to be. For f to be continuous, the approximation needs to get perfect as x gets close to a.

Example 1.14.

1. If $f(x) = 3x$ then $\lim_{x\to 1} f(x) = 3$.
2. If $f(x) = x^2$ then $\lim_{x\to 0} f(x) = 0$.
3. If $f(x) = \frac{x^2-1}{x-1}$ then

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1} x + 1 = 2.$$

Example 1.15. If $f(x) = \frac{x-1}{x^2-1}$ then what is $\lim_{x\to 1} f(x)$?

Answer: 1/2. If $y \neq 1$, then

$$f(y) = \frac{y-1}{(y-1)(y+1)} = \frac{1}{y+1}.$$

We know that $\frac{1}{x+1}$ is continuous, and that it is defined at $a = 1$. Thus $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$.

Example 1.16. $\lim_{x\to -2} \frac{(x+1)^2-1}{x+2} = \lim_{x\to -2} \frac{x^2+2x+1-1}{x+1} = \lim_{x\to -2} \frac{x(x+2)}{x+2} = \lim_{x\to -2} x = -2.$

Note that $\frac{x(x+2)}{x+2} \neq x$, but their limits at 0 are the same because the functions are the same near 0 (and in fact everywhere except at 0).

Example 1.17. What is $\lim_{x\to 0} \frac{\sqrt{9+x}-3}{x}$?

We use a trick called multiplication by the conjugate, which takes advantage of the fact that $(a+b)(a-b) = a^2 - b^2$. This trick is used *very often* so you should get comfortable with it.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{9+x}-3}{x} &= \lim_{x\to 0} \frac{\sqrt{9+x}-3}{x} \frac{\sqrt{9+x}+3}{\sqrt{9+x}+3} \\
&= \lim_{x\to 0} \frac{(9+x)-3}{x(\sqrt{9+x}+3)} = \lim_{x\to 0} \frac{x}{x(\sqrt{9+x}+3)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{9+x}+3} = \frac{1}{\lim_{x\to 0} \sqrt{9+x}+3} = \frac{1}{6}.
\end{aligned}$$

Example 1.18. What is $\lim_{x\to 1} \frac{x-1}{\sqrt{5-x}-2}$?

$$\begin{aligned}
\lim_{x\to 1} \frac{x-1}{\sqrt{5-x}-2} &= \lim_{x\to 1} \frac{x-1}{\sqrt{5-x}-2} \frac{\sqrt{5-x}+2}{\sqrt{5-x}+2} \\
&= \lim_{x\to 1} \frac{(x-1)(\sqrt{5-x}+2)}{(5-x)-4} \\
&= \lim_{x\to 1} \frac{(x-1)(\sqrt{5-x}+2)}{-(x-1)} \\
&= \lim_{x\to 1} -(\sqrt{5-x}+2) = -4.
\end{aligned}$$

Example 1.19. The *Heaviside Function* or *step function* is given by

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

It is often used in electrical engineering applications to describe the current running through a switch before and after it has been flipped.

We can ask: what is $\lim_{x\to 0} H(x)$?

There isn't one: no matter how close x gets to 0, sometimes $H(x)$ will be 0 and sometimes it will be 1. So there is no one value that approximates $H(x)$ for any x near a.

However, the Heaviside function clearly behaves well if look only at one side or the other of it. And just as we could talk about continuity to one side or the other, we can talk about *one-sided limits.*

Definition 1.20. Suppose a is a real number, and f is a function which is defined for all $x < a$ that are "near" the number a. We say "The limit of $f(x)$ as x approaches a from the left is L," and we write

$$\lim_{x\to a^-} f(x) = L,$$

if we can make $f(x)$ get as close as we want to L by picking x that are very close to (but less than) a.

Suppose a is a real number, and f is a function which is defined for all $x > a$ that are "near" the number a. We say "The limit of $f(x)$ as x approaches a from the right is L," and we write

$$\lim_{x \to a^+} f(x) = L,$$

if we can make $f(x)$ get as close as we want to L by picking x that are very close to (but greater than) a.

Under this definition, we see that $\lim_{x \to 0^-} H(x) = 0$ and $\lim_{x \to 0^+} H(x) = 1$.

Example 1.21. What is $\lim_{x \to 1^-} f(x)$ if $f(x) = \begin{cases} x^2 + 2 & x > 1 \\ x - 3 & x < 1 \end{cases}$?

Answer: -2.

1.3 Limits and Continuity

Now that we've defined limits and gotten a precise definition of continuity, we can discuss it in more depth.

The definition of continuity says that $\lim_{x \to a} f(x) = f(a)$. This secretly actually requires three distinct things to happen:

1. The function is defined at a; that is, a is in the domain of f.
2. $\lim_{x \to a} f(x)$ exists.
3. The two numbers are the same.

There are a few different ways for a function to be discontinuous at a point:

1. A function f has a *removable discontinuity* at a if $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$.
2. A function f has a *jump discontinuity* at a if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist but are unequal.
3. A function f has a *infinite discontinuity* if f takes on aribtrarily large or small values near a. We'll talk about this more soon.
4. It's also possible for the one-sided limits to not exist, but this doesn't have a special name. We saw this with $\sin(1/x)$. In this class, I'll call a function like this *really bad.*

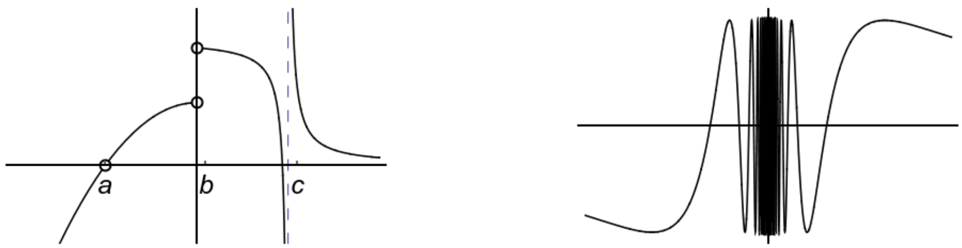


Figure 1.3: We saw this picture in section 1.1, but now we have language to talk about it.

Example 1.22. The Heaviside function of example 1.19 is not continuous, since there's a jump at 0.

It is continuous from the right at 0, since $\lim_{x \to 0^+} H(x) = 1 = H(0)$. This function is not continuous from the left, since $\lim_{x \to 0^-} H(x) = 0 \neq H(0)$.

Definition 1.23. A function is *continuous from the right at a* if $\lim_{x \to a^+} f(x) = f(a)$.

A function is *continuous from the left at a* if $\lim_{x \to a^-} f(x) = f(a)$.

Proposition 1.24. *A function is continuous at a if and only if it is continuous from the left and from the right at a.*

Remark 1.25. At a jump discontinuity, a function will often be continuous from one side but not the other. This is not necessarily the case, though: consider the function

$$f(x) = \begin{cases} 2 & x > 0 \\ 1 & x = 0 \\ 0 & x < 0 \end{cases}$$

Limits exist from the right and the left, but the function is not continuous from either side.

Recall we like continuous functions because we can use their values at one point to approximate the values they should have at nearby points. And we observed that this is really unhelpful at any point where the function isn't defined. So if we have a function that's continuous everywhere it's defined, we'd like to replace it with a function that is continuous—and defined—everywhere.

Definition 1.26. We say that g is an *extension* of f if the domain of g contains the domain of f, and $g(x) = f(x)$ whenever $f(x)$ is defined.

In general, we can only extend a function to be continuous at all real numbers if the only discontinuities were removable. This is why we call discontinuities like that "removable".

Example 1.27. Let $f(x) = \frac{x^2-1}{x-1}$. Can we define a function g that agrees with f on its domain, and is continuous at all reals?

f is continuous everywhere on its domain, and is undefined at $x = 1$. We can see that $g(x) = x + 1$ will give the same value as f everywhere on f's domain, and it is continuous since it is a polynomial. Thus g is a continuous extension of f to all reals.

Alternatively, we could compute that $\lim_{x\to 1} f(x) = 2$. Then we define

$$h(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ 2 & x = 1. \end{cases}$$

The function $h(x)$ is defined at all reals, and since it is continuous at 1 by our computation, it is continuous everywhere. It also must extend f since it is just defined to be f everywhere in the domain of f. So h is a continuous extension of f to all reals.

Importantly, g and h are actually the same function, since they give the same output for every input. There is at most one continuous extension of any given function; but there are multiple ways to describe that extension.

Example 1.28. The function $f(x) = 1/x$ is continuous on its domain, but we cannot extend it to a function continuous at all reals, because the limit at 0 does not exist.

Example 1.29. Let $f(x) = \frac{x^2-4x+3}{x-3}$. Can we extend f to a function continuous at all reals?

Answer: f is continuous at all reals except $x = 3$. But the function $g(x) = x - 1$ is the same everywhere except for 3, and is continuous at 3.

Example 1.30. Let

$$g(x) = \begin{cases} x^2 + 1 & x > 2 \\ 9 - 2x & x < 2 \end{cases}$$

Can we extend this to a continuous function on all reals?

Answer: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 9 - 2x = 5$, and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 + 1 = 5$, so the limit at 2 exists. Thus we can extend g to

$$g_f(x) = \begin{cases} x^2 + 1 & x \geq 2 \\ 9 - 2x & x \leq 2 \end{cases}$$

which is continuous at all reals.

1.4 Trigonometry and the Squeeze Theorem

We now want to look at limits of trigonometric functions. Fortunately, they behave *mostly* how we want them to.

Proposition 1.31. *If a is a real number, then* $\lim_{x\to a} \sin(x) = \sin(a)$ *and* $\lim_{x\to a} \cos(x) = \cos(a)$.

Proof. □

In fact, since trigonometric functions are just ways of combining sine and cosine, essentially all trigonometric functions behave this way where they are defined.

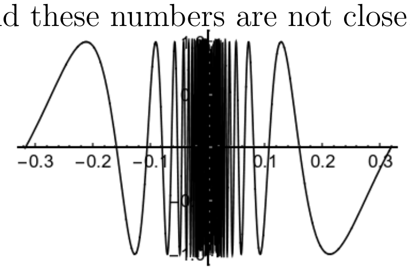
Example 1.32. $\lim_{x\to\pi} \cos(x) = -1$.

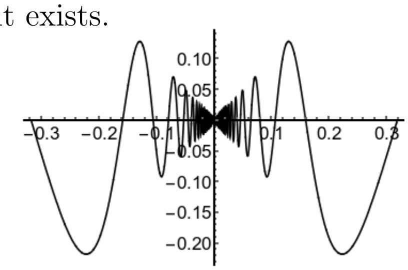
$\lim_{x\to\pi} \tan(x) = 0$.

But where the functions are not defined, sometimes very odd things can happen. We've seen a graph of $\sin(1/x)$ before, in section 1.1. We said that the function wasn't continuous at 0. In fact, no limit exists there.

Suppose a limit does exist at zero; specifically, let's suppose that $\lim_{x\to 0} \sin(1/x) = L$. Then if x is close to 0, it must be the case that $\sin(1/x)$ is close to L.

But however close we want x to be to 0, we can find a $x_1 = \frac{1}{(2n+1/2)\pi}$, and then $\sin(1/x_1) = \sin((2n + 1/2)\pi) = \sin(\pi/2) = 1$. But we can also find an $x_2 = \frac{1}{(2n+3/2)\pi}$ so that $\sin(1/x_2) = \sin(2n\pi + 3\pi/2) = \sin(3\pi/2) = -1$. So L must be really close to 1 and really close to -1, and these numbers are not close. So no limit exists.





Left: graph of $\sin(1/x)$, Right: graph of $x\sin(1/x)$

In contrast, from the graph it appears that $\lim_{x\to 0} x\sin(1/x)$ does exist. We can't possibly prove this by replacing $x\sin(1/x)$ with an almost identical function and plugging values in: the function is gross and complicated, and any almost identical function will also be gross and complicated.

But we can easily see that $\lim_{x\to 0} x = 0$. This doesn't mean that $\lim_{x\to 0} xf(x) = 0$ for any $f(x)$; if $f(x)$ gets really big then it can "cancel out" the x term getting very small. (A good example of this is $\lim_{x\to 0} x\frac{1}{x}$, which is of course 1).

But if we can prove that the second term, which in this case is $\sin(1/x)$, does *not* get really big, then the entire limit will have to go to zero. We make this intuition precise with the following important theorem:

Theorem 1.33 (Squeeze Theorem)**.** *If $f(x) \leq g(x) \leq h(x)$ near a (except possibly at a), and* $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, *then* $\lim_{x\to a} g(x) = L$.

To use the Squeeze Theorem, we need to do two things:

1. Find a lower bound and an upper bound for the function we're interested in; and
2. show that their limits are equal.

We usually do this by factoring the function we care about into two pieces, where one goes to zero and the other is bounded, and thus doesn't get infinitely big.

In this case, we know that $-1 \leq \sin(1/x) \leq 1$ by properties of $\sin(x)$. We "want" to multiply both sides of the equation by x to get $-x \leq x\sin(1/x) \leq x$, but this is actually incorrect when x is negative. In general, it's hard to reason about inequalities when negative numbers are involved, so we use absolute values to make sure we don't have to worry about it:

$$-|x| \leq x\sin(1/x) \leq |x|$$

Then we can compute that $\lim_{x\to 0}(-|x|) = \lim_{x\to 0}|x| = 0$ and so by the squeeze theorem, $\lim_{x\to 0} x\sin(1/x) = 0$.

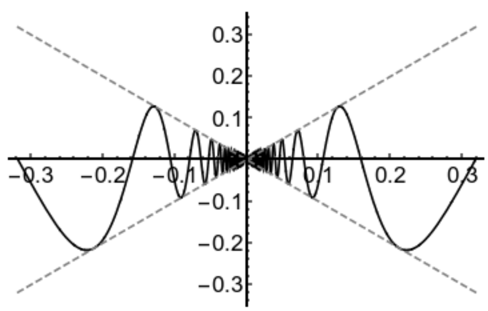


Figure 1.4: A graph of $x\sin(1/x)$ with $|x|$ and $-|x|$

This means that we can extend the function $x\sin(1/x)$ to be continuous at all reals, by defining

$$f(x) = \begin{cases} x\sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Remark 1.34. There is an argument people make sometimes that looks like the squeeze theorem, but is actually wrong. People reason:

$$\begin{aligned} -|x| &\leq x\sin(1/x) \leq |x| \\ \lim_{x\to 0} -|x| &\leq \lim_{x\to 0} x\sin(1/x) \leq \lim_{x\to 0}|x| \\ 0 &\leq \lim_{x\to 0} x\sin(1/x) \leq 0 \end{aligned}$$

and conclude that $\lim_{x\to 0} x\sin(1/x) = 0$.

However, this reasoning only works if you already know the limit exists. Compare:

$$\begin{aligned} -1 &\leq \sin(1/x) \leq 1 \\ \lim_{x\to 0} -1 &\leq \lim_{x\to 0} \sin(1/x) \leq \lim_{x\to 0} 1 \\ -1 &\leq \lim_{x\to 0} \sin(1/x) \leq 1. \end{aligned}$$

This uses the same reasoning, but the third statement doesn't actually make any sense because the limit doesn't exist. (Imagine writing that $-1 \leq \text{green} \leq 1$, for instance).

Example 1.35. Using the Squeeze Theorem, show that $\lim_{x\to 3}(x-3)\frac{x^2}{x^2+1} = 0$.

We could in fact do this without the squeeze theorem, but we also can use squeeze.

We divide the function into two parts. We see that $(x-3)$ approaches zero, so we need to bound the other factor.

We know that $0 \leq x^2 \leq x^2+1$ and so $0 \leq \frac{x^2}{x^2+1} \leq 1$ for any x. We want to multiply through by $x-3$, but that only works if $x > 3$. So we use absolute values to keep everything correct and get

$$0 \leq \left|(x-3)\frac{x^2}{x^2+1}\right| \leq |x-3|.$$

Then $\lim_{x\to 3} 0 = \lim_{x\to 3} -|x-3| = 0$, and so by the squeeze theorem $\lim_{x\to 3}(x-3)\frac{x^2}{x^2+1} = 0$.

Example 1.36. What is

$$\lim_{x\to 1} \frac{x-1}{2+\sin\left(\frac{1}{x-1}\right)}?$$

The top goes to zero and the bottom is bounded, so this looks like a squeeze theorem problem. If you have trouble seeing this, it may help to rewrite the problem as

$$\lim_{x\to 1}(x-1)\frac{1}{2+\sin\left(\frac{1}{x-1}\right)}.$$

We know that $-1 \leq \sin\left(\frac{1}{x-1}\right) \leq 1$ and so $1 \leq 2+\sin\left(\frac{1}{x-1}\right) \leq 3$, and thus

$$\begin{aligned} 1 &\geq \frac{1}{2+\sin\left(\frac{1}{x-1}\right)} \geq \frac{1}{3} \\ |x-1| &\geq \frac{|x-1|}{2+\sin\left(\frac{1}{x-1}\right)} \geq \frac{|x-1|}{3} \\ |x-1| &\geq \left|\frac{x-1}{2+\sin\left(\frac{1}{x-1}\right)}\right| \geq \frac{|x-1|}{3} \end{aligned}$$

since the denominator is always positive. But $\lim_{x\to 1} |x-1| = \lim_{x\to 1} \frac{|x-1|}{3} = 0$, so by the squeeze theorem

$$\lim_{x\to 1} \frac{x-1}{2+\sin\left(\frac{1}{x-1}\right)} = 0.$$

Example 1.37. Prove that $\lim_{x\to 3}(x-3)\left(5\sin\left(\frac{1}{x-3}\right)-2\right) = 0$.

We know that

$$\begin{aligned} -1 &\leq \sin\left(\frac{1}{x-3}\right) \leq 1 \\ -5 &\leq 5\sin\left(\frac{1}{x-3}\right) \leq 5 \\ -7 &\leq 5\sin\left(\frac{1}{x-3}\right) - 2 \leq 3. \end{aligned}$$

We want to multiply through by $x-3$, but this causes problems when $x < 3$ and thus $x - 3 < 0$. So first we put absolute values on everything.

But there's a subtlety here. We know our bad term is between -7 and 3. But when we take absolute values, that doesn't make it larger than $|-7|$ and smaller than $|3|$—no numbers satisfy those rules. Instead, we know that since we've added absolute values, everything will be bigger than zero. This gives us a lower bound.

For the upper bound, we care about how far away from zero we can get. One way to see this is that if $5\sin\left(\frac{1}{x-3}\right)-2 > 0$, we know that it must be less than 3; but if $5\sin\left(\frac{1}{x-3}\right)-2 < 0$, we know it must be bigger than -7, so the absolute value is < 7. So overall we get the bounds

$$0 \leq \left|(x-3)\left(5\sin\left(\frac{1}{x-3}\right)-2\right)\right| \leq |7(x-3)|.$$

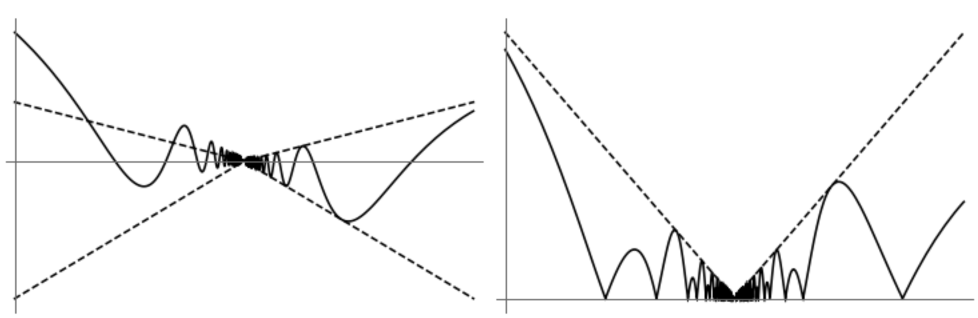


Figure 1.5: Left: $-7|x-3|$ is a fine lower bound, but $3|x-3|$ isn't an upper bound. Right: After we take absolute values, we see that $7|x-3|$ has the smallest coefficient we could possibly use and still get an upper bound.

Now we can compute that $\lim_{x\to 3} 0 = 0$ and $\lim_{x\to 3} |7(x-3)| = 0$, so by the squeeze theorem we know that $\lim_{x\to 3}(x-3)\left(5\sin\left(\frac{1}{x-3}\right)\right) = 0$.

Example 1.38. What is $\lim_{x\to -1}(x+1)\cos\left(\frac{x^5-3x^2+e^x-1700+(2+x)^{(1+x)^x}}{(x+1)^{27.2}}\right)$?

This looks complicated but is actually quite simple. $-1 \leq \cos(y) \leq 1$ for any y, including $y = x^5 - 3x^2 + e^x - 1700 + x^{x^x}$. Thus we have

$$0 \leq |\cos(y)| \leq 1$$
$$0 \leq |(x+1)\cos(y)| \leq |x+1|.$$

Then we know that $\lim_{x\to-1} 0 = \lim_{x\to-1}|x+1| = 0$. Thus by the squeeze theorem,

$$\lim_{x\to-1} |(x+1)\cos(x^5 - 3x^2 + e^x - 1700 + x^{x^x})| = 0,$$

and thus

$$\lim_{x\to-1} (x+1)\cos(x^5 - 3x^2 + e^x - 1700 + x^{x^x}) = 0.$$

Example 1.39. What is

$$\lim_{x\to 0} \frac{x-1}{2+\sin\left(\frac{1}{x-1}\right)}?$$

This is a trick question. Here we have no concerns about zeroes in the denominator or points outside of the domain, we can repeatedly apply limit laws:

$$\begin{aligned}\lim_{x\to 0} \frac{x-1}{2+\sin\left(\frac{1}{x-1}\right)} &= \frac{\lim_{x\to 0}(x-1)}{\lim_{x\to 0} 2+\sin\left(\frac{1}{x-1}\right)} \\ &= \frac{-1}{2+\sin\left(\lim_{x\to 0}\frac{1}{x-1}\right)} \\ &= \frac{-1}{2+\sin(-1)} = \frac{-1}{2-\sin(1)}.\end{aligned}$$

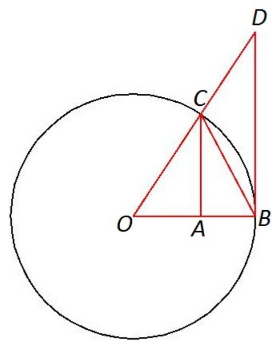
Remark 1.40. Notice that we don't conclude that since $f(x) \leq g(x) \leq h(x)$ then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x) \leq \lim_{x\to a} h(x)$. This is in fact not always true; it's only true if the middle limit exists, which is what we're trying to prove! So we just compute the outer two limits, and then invoke the squeeze theorem.

Example 1.41. $\lim_{x\to+\infty} \frac{\sin(x)}{x}$ exists, by the squeeze theorem.

For large x we have $\frac{-1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$, and $\lim_{x\to+\infty} \frac{-1}{x} = \lim_{x\to+\infty}\frac{1}{x} = 0$. So by the squeeze theorem $\lim_{x\to+\infty}\frac{\sin(x)}{x} = 0$.

You might notice this is *exactly the same proof* we gave for $\lim_{x\to 0} x\sin(1/x)$. This is not a coincidence, since the two functions are the same after the substitution $y = 1/x$.

There is one more important limit involving sin:



Proposition 1.42 (Small Angle Approximation)**.**

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Proof. We'll assume x is small and positive; this all still works if x is small and negative, with different signs. Our diagram is of a circle with radius 1.

Let x be the measure of angle AOC in our diagram. Observe that $\sin x$ is precisely the length of the line segment AC by definition, and so triangle BOC has area $\sin x/2$. The area of the entire circle is π and so the area of the wedge from B to C is $\pi x/2\pi = x/2$. Since the triangle is contained in the wedge, we have $\sin x/2 \leq x/2$ and thus $\sin x/x \leq 1$.

Note that AC is $\sin x$ and AO is $\cos x$, so AC over AO is $\sin(x)/\cos(x) = \tan(x)$. By similarity, we have DB $= \tan x$, and the area of triangle BOD is $\tan x/2$. Since the wedge from B to C is contained in this triangle, we have $x/2 \leq \tan x/2$ and thus $\cos x \leq \sin x/x$.

Thus $\cos x \leq \frac{\sin x}{x} \leq 1$. But $\lim_{x \to 0} \cos x = 1$, so by the squeeze theorem we have

$$1 \leq \lim_{x \to 0} \frac{\sin x}{x} \leq 1$$

and thus get the desired result. □

Remark 1.43. This means that the function

$$f(x) = \begin{cases} \sin(x)/x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

is a continuous extension of $\sin(x)/x$ to all reals.

Example 1.44. $\lim_{x \to 0} \frac{\sin(2x)}{2x} = 1$.

Example 1.45. What is $\lim_{x \to 0} \frac{\sin(4x)\sin(6x)}{\sin(2x)x}$?

We can write

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(4x)\sin(6x)}{\sin(2x)x} &= \lim_{x\to 0}\frac{\sin(4x)/4x\cdot\sin(6x)/6x\cdot 24x^2}{\sin(2x)/2x\cdot 2x\cdot x}\\ &= \lim_{x\to 0}\frac{\sin 4x}{4x}\cdot\frac{\sin 6x}{6x}\cdot\frac{2x}{\sin(2x)}\cdot\frac{24x^2}{2x^2}\\ &= 1\cdot 1\cdot 1\cdot 12 = 12.\end{aligned}$$

Here we are simply pairing off the $\sin(y)$'s with ys and then collecting the remainder into the last term.

Example 1.46. What is $\lim_{x\to 0}\frac{x}{\cos(x)}$?

This problem is actually easy. We can just plug in 0 for x and get $\lim_{x\to 0}\frac{x}{\cos(x)} = \frac{0}{1} = 0$.

In contrast, $\lim_{x\to 0}\frac{\cos(x)}{x}$ is mildly tricky, and we're not ready to do it yet. We'll discuss this sort of limit in section 1.5.1.

Example 1.47. What is $\lim_{x\to 0}\frac{x\sin(2x)}{\tan(3x)}$?

When we see a tangent in a problem, it is often helpful to rewrite it in terms of sin and cos. We can then collect terms:

$$\begin{aligned}\lim_{x\to 0}\frac{x\sin(2x)}{\tan(3x)} &= \lim_{x\to 0}\frac{x\sin(2x)}{\sin(3x)/\cos(3x)}\\ &= \lim_{x\to 0}\frac{3x}{\sin 3x}\cdot\frac{\sin(2x)\cos(3x)}{3} = 1\cdot\frac{0}{3} = 0.\end{aligned}$$

Example 1.48. What is $\lim_{x\to 3}\frac{\sin(x-3)}{x-3}$?

This is a small angle approximation again, since $x-3$ is approaching zero. Thus the limit is 1.

Example 1.49. What is $\lim_{x\to 3}\frac{\sin(x^2-9)}{x-3}$?

We have a $\sin(0)$ on the top and a 0 on the bottom, but the 0s don't come from the same form; we need to get a x^2-9 term on the bottom. Multiplication by the conjugate gives

$$\begin{aligned}\lim_{x\to 3}\frac{\sin(x^2-9)}{x-3} &= \lim_{x\to 3}\frac{\sin(x^2-9)}{x-3}\cdot\frac{x+3}{x+3} = \lim_{x\to 3}\frac{\sin(x^2-9)(x+3)}{x^2-9}\\ &= \lim_{x\to 3}\frac{\sin(x^2-9)}{x^2-9}\cdot\lim_{x\to 3}x+3 = 1\cdot(3+3) = 6.\end{aligned}$$

Example 1.50. What is $\lim_{x\to 0}\frac{1-\cos x}{x}$?

We can see that the limits of the top and the bottom are both 0, so this is an indeterminate form. We can't use the small angle approximation directly because there is no sin here at all. But we can fix that by multiplying by the conjugate.

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos(x)}{1+\cos(x)} = \lim_{x\to 0} \frac{1-\cos^2(x)}{x(1+\cos(x))} = \lim_{x\to 0} \frac{\sin^2(x)}{x(1+\cos(x))}$$
$$= \lim_{x\to 0} \frac{\sin(x)}{1+\cos(x)} = \frac{0}{2} = 0.$$

1.5 Infinite Limits

A few times in the past couple sections we've talked about vertical asymptotes, or functions going to infinity. In this section we want to look at exactly what that means. Some limits deal with infinity as an output, and others deal with it as an input (or both).

Remark 1.51. Recall that infinity is not a number. Sometimes while dealing with infinite limits we might make statements that appear to treat infinity as a number. But it's not safe to treat ∞ like a true number and we will be careful of this fact.

1.5.1 Limits To Infinity

Definition 1.52. We write

$$\lim_{x\to a} f(x) = +\infty$$

to indicate that as x gets close to a, the values of $f(x)$ get arbitrarily large (and positive).

We write

$$\lim_{x\to a} f(x) = -\infty$$

to indicate that as x gets close to a, the values of $f(x)$ get arbitrarily negative.

We write

$$\lim_{x\to a} f(x) = \pm\infty$$

to indicate that as x gets close to a, the values of $f(x)$ get arbitrarily positive or negative. We usually use this when both occur.

Remark 1.53. Important note: If the limit of a function is infinity, the limit *does not exist.* This is utterly terrible English but I didn't make it up so I can't fix it. All the theorems that say "If a limit exists" are not including cases where the limit is infinite.

Lemma 1.54. *Let $f(x), g(x)$ be defined near a, such that $\lim_{x\to a} f(x) = c \neq 0$ and $\lim_{x\to a} g(x) = 0$. Then*

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \pm\infty.$$

Further, assuming $c > 0$ then the limit is $+\infty$ if and only if $g(x) \geq 0$ near a, and the limit is $-\infty$ if and only if $g(x) \leq 0$ near a. If $c < 0$ then the opposite is true.

Remark 1.55. If the limit of the numerator is zero, then this lemma is *not useful*. That is one of the "indeterminate forms" which requires more analysis before we can compute the limit completely.

Example 1.56. What is $\lim_{x\to 3} \frac{-1}{\sqrt{x-3}}$? We see the top goes to 1 and the bottom goes to 0, so the limit is $\pm\infty$. Since the denominator is always positive and the numerator is negative, the limit is $-\infty$.

We have to be careful while working these problems: the limit laws that work for finite limits don't always work here, since the limit laws assume that the limits exist, and these do not. In particular, adding and subtracting infinity *does not work*. Instead, we need to arrange the function into a form where we can use lemma 1.54.

Example 1.57. We already know that $\lim_{x\to 0} 1/x = \pm\infty$.

1. If we take $\lim_{x\to 0} 1/x - 1/x$, we could say the limit is $\pm\infty - \pm\infty$, but this is silly—the limit is actually 0.

2. In contrast, $\lim_{x\to 0} 1/x + 1/x = \lim_{x\to 0} 2/x = \pm\infty$. We don't add the infinities together.

3. And $\lim_{x\to 0} 1/x + 1/x^2$ is the trickiest. We have a $\pm\infty$ plus a $+\infty$. But again we can't add infinities—we need to combine them into one term.

   $$\lim_{x\to 0} \frac{1}{x} + \frac{1}{x^2} = \lim_{x\to 0} \frac{x+1}{x^2} = +\infty$$

   since the numerator approaches 1 and the denominator approaches 0, but is always positive.

   We could heuristically say that $\frac{1}{x^2}$ goes to $+\infty$ "faster" than $\frac{1}{x}$ goes to $\pm\infty$, and so it wins out; but this is really vague and handwavy so we try to replace it with more precise arguments like this one.

We organize our thinking about these situations in terms of the "indeterminate forms", which are: $\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty\pm\infty, 1^\infty, \infty^0$. Notice that none of these are actual numbers, and they can never be the correct answer to pretty much any question.

More importantly, indeterminate forms don't even tell us what the answer should be; if plugging in gives you one of those forms, the true limit could potentially be pretty much anything. We have to do more work to get our functional expression into a determinate form. As a general rule, we use algebraic manipulations to get a form of $\frac{0}{0}$, then factor out and cancel $(x-a)$ until either the numerator or the denominator is no longer 0.

Remark 1.58. Neither $\frac{0}{1}$ nor $\frac{1}{0}$ is an indeterminate form. $\frac{0}{1}$ is just a number, equal to 0. $\frac{1}{0}$ is not a number and is never the correct answer to a question, but it's also not indeterminate. By lemma 1.54, if $\lim f(x) = 1$ and $\lim g(x) = 0$ then $\lim f(x)/g(x) = \pm\infty$.

Similarly, $\frac{0}{\infty}$ and $\frac{\infty}{0}$ are also not numbers but not indeterminate. The first suggests the limit is 0; the second suggests the limit is $\pm\infty$.

The form $\infty \cdot \infty$ mostly works fine, and gives you another ∞ whose sign depends on the signs of the ∞s you're multiplying. But again, $\infty \cdot \infty$ is never the actual answer to any actual question.

Example 1.59. What is $\lim_{x\to-2} \frac{1}{x+2} + \frac{2}{x(x+2)}$? This looks like $\infty + \infty$ so we have to be careful. We have

$$\lim_{x\to-2} \frac{1}{x+2} + \frac{2}{x(x+2)} = \lim_{x\to-2} \frac{x}{x+2} + \frac{2}{x(x+2)}$$
$$= \lim_{x\to-2} \frac{x+2}{x(x+2)} = \lim_{x\to-2} \frac{1}{x} = \frac{-1}{2}.$$

Example 1.60. $\lim_{x\to3^+} \frac{1}{(x-3)^3} = +\infty$: the limit of the top is 1, and the limit of the bottom is 0, so the limit is $\pm\infty$. But when $x > 3$ the denominator is ≥ 0, so the limit is in fact $+\infty$. Conversely $\lim_{x\to3^-} \frac{1}{(x-3)^3} = -\infty$ since when $x < 3$ we have $(x-3)^3 \leq 0$.

$\lim_{x\to-1^+} \frac{1}{(x+1)^4} = +\infty$. And $\lim_{x\to-1^-} \frac{1}{(x+1)^4} = +\infty$. Thus $\lim_{x\to-1} \frac{1}{(x+1)^4} = +\infty$.

1.5.2 Limits at infinity

A related concept is the idea of limits "at" infinity, which answers the question "what happens to $f(x)$ when x gets very big?" We can formally define this in terms of ϵ.

Definition 1.61. Let f be a function defined for (a, ∞) for some number a. We write

$$\lim_{x\to+\infty} f(x) = L$$

to indicate that when x is large enough, the values of $f(x)$ get arbitrarily close to L. Formally, if for every $\epsilon > 0$ there is a $M > 0$ so that if $x > M$ then $|f(x) - L| < \epsilon$.

We can write similar definitions for $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\pm\infty} f(x)$, and talk about when these limits are themselves $\pm\infty$. But here we'll skip over the formal definition and simply think informally.

In principle, we want to do the same thing we did for finite limits. But instead of having zeros on the top and bottom of a fraction, we often have infinities as well. So we want to "cancel" an infinity from the top and the bottom of the fraction. We usually do this by dividing the top and bottom by x. Then we can use the following crucial fact:

Fact 1.62. $\lim_{x\to\pm\infty} \frac{1}{x} = 0$.

This combined with tools we already have is enough to do pretty much any calculuation here.

Example 1.63. If we want to calculuate $\lim_{x\to+\infty} \frac{1}{\sqrt{x}}$, we see that

$$\lim_{x\to+\infty} \frac{1}{\sqrt{x}} = \sqrt{\lim_{x\to+\infty} \frac{1}{x}} = \sqrt{0} = 0.$$

Example 1.64. What is $\lim_{x\to+\infty} \frac{x}{x^2+1}$?

This problem illustrates the primary technique we'll use to solve infinite limits problems. It's difficult to deal with problems that have variables in the numerator and denominator, so we want to get rid of at least one. Thus we will divide out by xs on the top and the bottom until one has none left:

$$\lim_{x\to+\infty} \frac{x}{x^2+1} = \lim_{x\to+\infty} \frac{x/x}{x^2/x + 1/x} = \lim_{x\to+\infty} \frac{1}{x+\frac{1}{x}} = \lim_{x\to+\infty} \frac{1}{x} = 0.$$

Example 1.65. Some more examples of this technique:

$$\lim_{x\to-\infty} \frac{x}{x+1} = \lim_{x\to-\infty} \frac{1}{1+\frac{1}{x}} = \lim_{x\to-\infty} \frac{1}{1} = 1.$$

$$\lim_{x\to-\infty} \frac{x}{3x+1} = \lim_{x\to-\infty} \frac{1}{3+\frac{1}{x}} = \frac{1}{3}.$$

Example 1.66. What is $\lim_{x\to+\infty} \frac{x^{3/2}}{\sqrt{9x^3+1}}$? This one is a bit tricky. We want to divide the top and bottom by $x^{3/2}$. Then we can pull the factor *inside* the square root sign.

$$\lim_{x\to+\infty} \frac{x^{3/2}}{\sqrt{9x^3+1}} = \lim_{x\to+\infty} \frac{1}{\sqrt{9+1/x^{3/2}}} = \frac{1}{\sqrt{9+0}} = \frac{1}{3}.$$

Example 1.67. Sometimes it's a bit harder to see how this works. For instance, what is $\lim_{x\to+\infty} \frac{x}{\sqrt{x^2+1}}$? It's not obvious, but we use the same technique:

$$\lim_{x\to+\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to+\infty} \frac{x/x}{\sqrt{x^2+1}/x} = \lim_{x\to+\infty} \frac{1}{\sqrt{x^2/x^2+1/x^2}} = \lim_{x\to+\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = 1.$$

Example 1.68. What is $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}}$?

We can do the same thing, but we have to be *very careful*. Remember that if $x < 0$ then $\sqrt{x^2} \neq x$! Instead, $x = -\sqrt{x^2}$. Thus we have

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{1}{\sqrt{x^2+1}/x} = \lim_{x\to-\infty} \frac{1}{\sqrt{x^2+1}/(-\sqrt{x^2})} = \lim_{x\to-\infty} \frac{1}{-\sqrt{1+\frac{1}{x^2}}} = -1.$$

When we encounter new functions, one of the ways we will often want to characterize them is by computing their limits at $\pm\infty$. Sometimes these limits do not exist.

Example 1.69. $\lim_{x\to+\infty}\sin(x)$ does not exist, since the function oscillates rather than settling down to one limit value.

$\lim_{x\to+\infty} x\sin(x)$ also does not exist; this function oscillates more and more wildly as x increases.

But $\lim_{x\to+\infty}\frac{1}{x}\sin(x)$ does in fact exist. We can prove this with the squeeze theorem: we can see that $\frac{-1}{x}\leq\frac{1}{x}\sin(x)\leq\frac{1}{x}$, and we know that $\lim_{x\to+\infty}\frac{-1}{x}=\lim_{x\to+\infty}\frac{1}{x}=0$. So by the Squeeze Theorem, $\lim_{x\to+\infty}\frac{1}{x}\sin(x)=0$.

Another technique that will also often appear in these limits is combining a sum or difference into one fraction. If we have a sum of two terms that both have infinite limits, we need to combine or factor them into one term to see what is happening.

Example 1.70. What is $\lim_{x\to-\infty} x-x^3$?

Each term goes to $-\infty$, so this is a difference of infinities and thus indeterminate. But we can factor: $\lim_{x\to-\infty} x(1-x^2)$. The first term goes to $-\infty$ and the second term also goes to $-\infty$, so we expect that their product will go to $+\infty$. Thus $\lim_{x\to-\infty} x-x^3=+\infty$.

To be precise, I should compute:

$$\lim_{x\to-\infty} x-x^3=\lim_{x\to-\infty}\frac{x-x^3}{1}=\lim_{x\to-\infty}\frac{1/x^2-1}{1/x^3}.$$

We see the limit of the top is -1 and the limit of the bottom is 0, so the limit of the whole is $\pm\infty$. In fact the bottom will always be negative (since $x\to-\infty$), and thus the limit is $+\infty$.

Example 1.71. What is $\lim_{x\to+\infty}\sqrt{x^2+1}-x$?

We might want to try to use limit laws here, but we would get $+\infty - +\infty$ which is not defined (and is one of the classic indeterminate forms). Instead we need to combine our expressions into one big fraction.

$$\begin{aligned}\lim_{x\to+\infty}\sqrt{x^2+1}-x &= \lim_{x\to+\infty}\left(\sqrt{x^2+1}-x\right)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\\ &= \lim_{x\to+\infty}\frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x}=\lim_{x\to+\infty}\frac{1}{\sqrt{x^2+1}+x}.\end{aligned}$$

The numerator is 1 and the denominator approaches $+\infty$ so the limit is 0. This tells us that as x increases, x and $\sqrt{x^2+1}$ get as close together as we wish.

You may have noticed the appearance of our old friend, multiplication by the conjugate. We will often use that technique in this sort of problem.

Example 1.72. What is $\lim_{x \to +\infty} \sqrt{x^2+x+1} - x$?

$$\begin{aligned}
\lim_{x \to +\infty} \sqrt{x^2+x+1} - x &= \lim_{x \to +\infty} \left(\sqrt{x^2+x+1} - x\right) \frac{\sqrt{x^2+x+1}+x}{\sqrt{x^2+x+1}+x} \\
&= \lim_{x \to +\infty} \frac{x^2+x+1-x^2}{\sqrt{x^2+x+1}+x} = \lim_{x \to +\infty} \frac{x+1}{\sqrt{x^2+x+1}+x} \\
&= \lim_{x \to +\infty} \frac{1+1/x}{\sqrt{1+1/x+1/x^2}+1} = \frac{1}{2}.
\end{aligned}$$