3 Models: Relationships and Differential Equations

So far we've been using math to study math. We can use derivatives to approximate functions, but functions are themselves sort of purely mathematical things. In this section we want to see how we can use our mathematical tools to study and model the actual world, and understand the physical meaning of the derivative and other mathematical tools we have learned.

3.1 Rates of Change

In section 2 we developed the idea of the derivative in two different ways. First we took an algebraic approach. We looked for a way to approximate a function with a line; we took the approximation $f(x) \approx f(a) + f'(a)(x - a)$ and asked what number the slope $f'(a)$ should be. From this we derived the definition of the derivative.

Separately, we took a geometric approach. We drew the curve corresponding to the graph of a function, and asked for the line tangent to that curve at a given point. We drew secant lines through two nearby points, and as the points got closer together we found the tangent line. From this we got the same formula we got from linear approximation, which is the definition of derivative.

Now we want to approach the derivative a third way, from a physical perspective. We'll see that the derivative represents the rate at which some quantity changes.

We often start by considering the idea of *speed*. Speed is defined to be distance covered divided by time spent; that is, $v = \frac{\Delta x}{\Delta t}$. In particular, if your position at time t is given by the function $p(t)$, then your average speed between time t_0 and time t_1 is

$$v = \frac{p(t_1) - p(t_0)}{t_1 - t_0}.$$

This formula should look familiar. It is the slope of a line through the points $(t_0, p(t_0))$ and $(t_1, p(t_1))$. It is *not* the derivative of p, because we didn't take a limit. It is instead a "difference quotient", which is really a fancy way of saying the slope of a line.

Example 3.1. For example, on Earth dropped objects fall about $p(t) = 5t^2$ meters after t seconds. The average speed between time $t = 1$ and time $t = 2$ is

$$v = \frac{p(2) - p(1)}{2 - 1} = \frac{20 - 5}{1} = 15\text{m/s}$$

and the average speed between time $t = 3$ and time $t = 1$ is

$$v = \frac{p(3) - f(1)}{3 - 1} = \frac{45 - 5}{3 - 1} = 20\text{m/s}.$$

It's useful here to look at the units. We know that the result is a speed, so comes out in m/s. But how do we know we get those units? We have to think a bit about what the function p is actually doing.

The function p gives us position as a function of time. Thus the *inputs* to p are given in seconds, and the *outputs* are given in meters. So it's not really fully correct to say that $p(t) = 5t^2$; that would suggest that $p(1\text{s}) = 5(1\text{s})^2 = 5\text{s}^2$. But your position isn't described in square seconds!

Instead, we would write something like $p(t\text{second}) = 5t^2\text{m}$. The function takes in seconds as inputs, and gives meters as outputs. Thus our last calculation properly should have been

$$v = \frac{p(3\text{s}) - f(1\text{s})}{3\text{s} - 1\text{s}} = \frac{45\text{m} - 5\text{m}}{3\text{s} - 1\text{s}} = 20\text{m/s}.$$

We see that the numerator—which is made up of the outputs of p—has units of meters, while the denominator, which is made up of the inputs of p, has units of seconds. So the entire fraction has units of m/s, which is what it should be.

We can give a more general formula. What's the average speed between time $t_0 = 1$ and time $t_1 = t$? We have

$$v = \frac{p(t\text{s}) - p(1\text{s})}{t\text{s} - 1\text{s}} = \frac{5t^2\text{m} - 5\text{m}}{t\text{s} - 1\text{s}} = 5(t+1)\frac{t-1}{t-1}\text{m/s}.$$

As long as $t \neq 1$, this gives us a formula for average speed between time t and time 1: the average speed is $5(t+1)$m/s. But what if we want to know the speed "at" the time $t = 1$?

On some level, this question doesn't make any sense. Speed is defined as the change in distance divided by the change in time; if time doesn't change, and distance doesn't change, then this doesn't really mean anything. But we'd like it to mean something, so we take a limit instead.

Thus we can define your *instantaneous speed* or *speed at time* t_0 to be

$$\lim_{t_1 \to t_0} \frac{p(t_1) - p(t_0)}{t_1 - t_0} = \lim_{h \to t_0} \frac{p(t_0 + h) - p(t_0)}{h}.$$

Notice that since the function p has input in seconds and output in meters, the instantaneous speed will be in m/s, as it should be. But also notice that this formula is just the definition of the derivative of p.

Thus from the previous example, we can see that the instantaneous speed at time $t_0 = 1$ is

$$v(1\text{s}) = p'(1\text{s}) = \lim_{t \to 1} 5(t+1)\frac{t-1}{t-1}\text{m/s} = 10\text{m/s}.$$

Alternatively, we know that $p(t) = 5t^2$, so by our derivative rules we know that $p'(t) = 10t$ and thus $p'(1) = 10$. Once we add units, we have $p'(t\text{s}) = 10t\text{m/s}$ and thus $p'(1\text{s}) = 10\text{m/s}$.

Thus the derivative of a function has different units from the original function. Since the derivative is given by a formula with output in the numerator and input in the denominator, the derivative will have the units of the output per units of input.

We can take this one step further and look at the derivative of p'. The function p' takes in a time and outputs a speed; its derivative will be

$$p''(t_0\text{s}) = \lim_{t \to t_0} \frac{p'(t\text{s}) - p'(t_0\text{s})}{t\text{s} - t_0\text{s}}.$$

The units of the denominator are still seconds; but the units of the top are m/s, so the second derivative takes in seconds and outputs meters per second *per second*, or m/s^2. This makes sense: the second derivative is the change in the first derivative, so p'' tells us how quickly the speed is changing. So it tells us how many meters per second your speed changes each second.

The derivative occurs in lots of other situations that aren't just physical speed. One common place they show up is in economics.

Example 3.2 (Debt and Deficit)**.** A lot of discussions of economics and public policy address the deficit and the debt. The "deficit" and the "debt" are easy to confuse but importantly different, in a way that maps cleanly to the idea of a derivative.

A "deficit" is the amount of money that is currently owed; it is measured in dollars (or euro or yen or some other currency). The current US national deficit is approximately \$22 trillion.

A "deficit" is the rate at which the debt is increasing. So the national deficit is currently about \$1 trillion. This means we expect the debt next year to be about \$1 trillion bigger than the debt this year.

Mathematically we can define a function $D(t)$ which takes in the year and outputs the number of dollars owed. Then the annual deficit is

$$\frac{D((t+1)\text{y}) - D(t\text{y})}{1\text{y}}.$$

This isn't a derivative, since there's no limit; this is a *difference quotient* that measures a discrete change in debt over a discrete time. It's analogous to average speed, not instantaneous speed.

But we could imagine asking how the deficit is changing from month to month, or from week to week, or from hour to hour. We can take a limit as the time between $t + h$ and t

goes to zero, and then the deficit would be the derivative of debt. The function $D'(t)$ will take in years, and output dollars per year.

What about the second derivative? The function D'' will take in years, and output the yearly change in the deficit, measured in dollars per year per year. When people talk about whether the deficit is going up or down, they are looking at the second derivative of the debt.

Both of these examples have one very important trait in common. The position function $p(t)$ and the debt function $D(t)$ output different types of things with different units, but they both take *time* as an input. But it's easy for a function to take inputs other than time, and these functions are often physically important and meaningful.

Example 3.3 (Slope)**.** We've already seen one example of this, in lab 4 when we studied tangent lines. If I'm thinking about the graph of a function, then the input to the function is a horizontal position, measured in inches (or some other unit of distance). And the output is a vertical position, also measured in inches. So $f(x)$ takes in inches and outputs inches.

The derivative $f'(x)$ will still take in inches. But if we compute the derivative $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, then the denominator is in inches and the numerator is also in inches. This makes the derivative technically unitelss—but in reality, it is measured in inches per inch.

And this has a clear physical interpretation! We already know the derivative gives us the slope of the tangent line, and the slope measures how many units the graph goes up for each unit it goes over. Thus, it measures inches of horizontal position per inch of vertical position.

The second derivative $f''(x)$ will take in inches and output 1/inch, which is really inches per inch per inch. It tells us how much the slope, measured in inches per inch, changes if we move one inch horizontally.

Example 3.4 (Price Elasticity of Demand)**.** Another common economics question is to see how the demand for a product relates to its price. We can define a function $Q(p)$ that takes in a price in dollars, and outputs the quantity of items that will be bought. So if $Q(p) = 10000 - 10p$, this means that if the price is \$100 then people will buy $Q(100) = 10000 - 1000 = 9000$ widgets.

What's the derivative here? The function $Q'(p)$ takes in a price in dollars and outputs a number of widgets per dollar. It tells you how the quantity demanded changes in response to changes in the price. Thus we see that since $Q'(p) = -10$, we expect to sell ten fewer widgets for each dollar we raise the price.

(Economists call this the Price Elasticity of Demand: "elasticity" is how quickly one thing responds to changes in another thing. So any time the term "elasticity" shows up in economics, there's a derivative inolved somewhere).

What if instead we had the function $Q(p) = 10000 - 5p^2$? Now we see that changing the price doesn't have a huge effect if the price is already small, but it has a dramatic effect if the price is big. We compute that $Q'(p) = -10p$. This means that increasing the price by one dollar will decrease the quantity demanded by ten widgets for every dollar of the price.

Thus if the current price is \$10, we expect raising the price to \$11 to reduce sales by about a hundred widgets. If the current price is \$30 then raising the price will lose us nine hundred widgets in sales.

Example 3.5 (Ohm's Law)**.** In physics and electrical engineering, Ohm's Law tells us that current is equal to voltage over resistance, or $I = V/R$. (Here current is generally measured in amperes, voltage in volts, and resistance in, essentially, volts per amp).

The default assumption in most physics problems is that resistance is constant, a property of whatever material you're putting current through. So we have the function $I(V) = \frac{1}{R}V$, which is a linear function and simple to work with.

But this is just an approximation! Most materials will actually have their resistance change as the voltage applied to them changes, so the equation above is just a linear approximation to the actual relationship between current and voltage. This means that the slope $\frac{1}{R}$ is really a derivative.

An incandescent lightbulb works by running a current through a metal wire until it heats up. But as the heat of the wire increases, the resistance goes up. Thus the graph of current as a function of voltage is curving down; the higher the voltage, the less extra current you get from adding another volt. This means that the derivative $\frac{dI}{dV}$ is large when V is small, but small when V is large.

A diode is a material that does the opposite. Resistance is high when the voltage is low, but past some transition point the resistance drops and becomes very low. This means that the derivative is large when V is small, and then small when V is large. The graph of I as a function of V will curve up.

In practice. engineers mostly don't want to worry about the whole curve. If they know about what voltage their devices will experience, they don't need to worry what happens in other places. So they take the local linear approximation, call that "the resistance", and use the equation $I = I_0 + \frac{1}{R}(V - V_0)$. And this is just the linear approximation equation we've been using all class.

3.2 Differential Equations

Now that we have assigned derivatives a physical meaning, or even a large number of physical meanings, we can reason about how they interact with physical systems. In particular, we can make statements about the derivative of some function and how it relates to the values of that function.

Definition 3.6. A *differential equation* is an equation that relates the derivatives of a function to the values of that function.

For a simple example, consider the phrase "acceleration is proportional to force." Recall that acceleration is the second derivative of position. If force is itself a function of position, this translates to a differential equation, relating $f''(x)$ to $f(x)$.

Example 3.7 (Hooke's Law)**.** Hooke's law tells us that the force a fall exerts is proportional to the displacement of the fall; that is, for any given fall there is some constant k such that $F(t) = -kx(t)$, where $x(t)$ is the function that takes in the time and outputs the x coordinate of the fall. Since $F(t) = ma(t) = mx''(t)$, this gives us the differential equation $mx''(t) = -kx(t)$ or

$$x''(t) = -\frac{k}{m}x(t).$$

For simplicity let's assume $k = m$ so we have $x''(t) = -x(t)$.

Can we find a solution for this? We can start with the really silly or "trivial" solution. If the fall starts at neutral, it will never move, so we'd expect $x(t) = 0$. And indeed it is: $0'' = 0 = -0$, so the funcion $x(t) = 0$ is a solution to this differential equation.

Can we find a solution that involves any motion at all? We're looking for a function where $x''(t) = -x(t)$. And we actually know two of these: $x(t) = \sin(t)$ and $x(t) = \cos(t)$ both satisfy this differential equation. And this is why the equation for "simple harmonic motion" is built up out of sin and cos functions.

There are many different solutions we can use; for example, $3\sin(t) + 5\cos(t) = 17$ is a solution to this differential equation. It's easy to see that if a and b are any constants, then $x(t) = a\sin(t) + b\cos(t)$ is a solution to this differential equation. It's much less obvious, but true, that any solution to the Hooke's Law equation must have this form; even the trivial solution is given by $x(t) = 0\sin(t) + 0\cos(t)$.

To pick out the specific solution we need to know "initial conditions" that tell us the starting position. But if we know the starting position and starting velocity of the fall, we can determine a and b and thus get an exact formula for $x(t)$.

Something very interesting has happened here! In grade school, we learned to do simple arithmetic, like being asked to compute $3 + 5$ and calculating 8. As we got to algebra, we were asked instead to *solve equations*. We would get formulas like $3 + x = 8$ and try to figure out what x is. This is the same sort of question but backwards—instead of computing with known numbers, we have to figure out which numbers will make the calculation work.

In pre-calculus and so far in this course, we have done calculuations with functions. Plug a number into this function; graph this function; take the derivative of this function. But here we are being asked to solve equations whose answers are *functions*. The question is, which function satisfies the given relationship? And if we have a candidate answer, we can test it by plugging it into the differential equation and seeing if the equation we get is true.

Example 3.8 (Proportional Growth)**.** The simplest possible (non-trivial) differential equation is probably $p'(t) = kp(t)$. This tells us that the rate of change of something we're measuring is proportional to the current level of that thing.

This often comes up in the context of population growth. If we look at, say, a breeding population of rabbits, then the number of new rabbits born each year depends on the number of rabbits that are already alive: if we start with two rabbits, we won't end the year with two million. If each rabbit on average produces three new rabbits in a year, we might approximate the derivative by saying $\frac{dp}{dt} = 3p(t)$. That is, the change in the total population of rabbits is equal to three times the current number of rabbits.

In this case, if we start a year with 100 rabbits, then we have $p'(0) = 3p(0) = 300$ so we expect to get three hundred new rabbits, and end the year with 400. The next year we will get $p'(1) = 3p(1) = 1200$, so we get 1200 new rabbits and end the year with 1600 rabbits. The derivative is different each year, but the proportional growth rate is not.

Can we find a function that satisfies $p'(t) = 3p(t)$? And so far in this course, the answer is "not really". The trivial solution will still work, actually: if we start with zero rabbits, then we will always have zero rabbits, and it is true that $0' = 3 \cdot 0$. But if we want a non-trivial solution, none of the functions we've seen so far will work here. In section 3.4 we will see that the solutions to this differential equation look like $p(t) = Ca^t$ for some constants C and a; a depends on the breeding rate, and C is the initial population of rabbits.

But this equation describes more than just rabbit population growth. Other cases where this equation appears include:

- Economic growth: the economy grows by 3% a year, so we have $p'(t) = .03p(t)$.
- Interest: if you are paying 8% interest per year, then your debt increases at a rate $d'(t) = .08d(t)$.

- Radioactive decay: some fraction of your sample of uranium will decay every year, so you have $u'(t) = ku(t)$. In this case k will be negative since your amount of uranium is decreasing.
- Heat transfer: the rate at which heat flows from a hot object to a cold object is proportional to the difference in temperature, so we have $T'(t) = kT(t)$.

Example 3.9 (Evans price change model)**.** Economists often use systems of differential equations to describe how the economy changes over time.

If there is a shortage of some good, which means that more people want to buy than sell, the price will tend to increase so that fewer people want to buy, more people want to sell, and the market clears. But the price doesn't change immediately. The Evans model says that the price change is proportional to the size of the shortage: $\frac{dp}{dt} = k(D - S)$, where D is the quantity demanded and S is the quantity supplied. So if the shortage is bigger, the price will increase faster.

So far, this looks sort of like exponential growth. But it's importantly different, because the size of the shortage is not the same thing as the price! We need to ask how demand depends on price. A simple model says that $D(p) = a - bp$ and $S(p) = r + sp$, where a is the amount demanded when the price is zero and r is the amount supplied when the price is zero. Then $-b = \frac{dD}{dp}$ and $s = \frac{dS}{dp}$ are the elasticities of demand and supply.

Plugging this back into the original model gives

$$p'(t) = k(a - bp(t) - r - sp(t)) = k\left((a - r) - (b + s)p(t)\right).$$

From this we can see that the trivial solution where the price is zero doesn't actually work here. And this should make sense, because if the price is zero you expect more people to want to buy than to sell. We also notice that it doesn't matter what the demand or supply elasticities are individually; it only matters what their sum is. We can use this equation to estimate the way the price will change over time.

There is a rich and powerful theory for solving differential equations. We won't really be studying it in this course, since we don't have the tools to understand it; we would need integrals from calculus 2 and also a number of tools from linear algebra. But there are a few questions we can address.

First, we can check whether a given function actaully satisfies a given differential equation.

Example 3.10. Confirm that $f(x) = x^2 + x + 1$ satisfies $2f(x) - xf'(x) = x + 2$.

We compute $f'(x) = 2x + 1$, so $2f(x) - xf'(x) = 2x^2 + 2x + 2 - (2x^2 + x) = x + 2$.

Second, we can solve what are called "initial value problems" or "boundary value problems". A given differential equation will usually have infinitely many solutions, as with the solutions $a\sin(t) + b\cos(t)$ to the equation $x''(t) = -x(t)$. This tells us the general shape of the solution, but doesn't give us an actual solution.

As we discussed, the specific solution depends on where things start. On the Hooke's Law fall system, if your fall starts at neutral then it will never move; if it starts extremely displaced then it will oscillate wildly. So to know the position over time we need to know where the system starts, known as the *initial conditions*.

Example 3.11. Suppose we have a Hooke's Law system with $m = k$, so that we get the differential equation $x''(t) = -x(t)$. We said earlier that then $x(t) = a\sin(t) + b\cos(t)$ for some constants a and b.

Suppose now we start with the weight stationary and displaced by 1 meter. Since this is the starting conditions, this is at time 0, so this means that $x(0) = 1$ and $x'(0) = 0$. Now we have enough information to figure out a and b and find a specific solution to describe the path of our fall.

Since $x(0) = 1$ we know that

$$1 = a\sin(0) + b\cos(0) = b,$$

and since $x'(0) = 0$ we know that

$$0 = a\cos(0) - b\sin(0) = a$$

so we have $a = 0, b = 1$, and $x(t) = \cos(t)$.

And as we think about it, this answer makes some sense: there's no reason for the fall to ever displace further than one meter, and so that's exactly what we see here.

Sometimes instead of an initial value problem we have a boundary value problem. In a boundary value problem you get position values at different times, rather than position and velocity at the same time.

Example 3.12. Suppose we have a Hooke's Law setup with $m-k$, so $x(t) = a\sin(t)+b\cos(t)$.

Suppose we know that $x(0) = 2$ and $x(\pi/4) = \sqrt{8}$. Then we know that

$$\begin{aligned} a\sin(0) + b\cos(0) &= 2 \\ b &= 2 \\ a\sin(\pi/4) + b\cos(\pi/4) &= \sqrt{8} \\ a\sqrt{2}/2 + 2 \cdot \sqrt{2}/2 &= \sqrt{8} \\ a/2 + 1 &= 2 \\ a &= 2. \end{aligned}$$

Thus we have that $x(t) = 2\sin(t) + 2\cos(t)$.

Notice that in either of these cases, we need to take only two measurements to know exactly what happens at every possible time. This is because our differential equation, coming from a physical law, severely constrains what our answers can possibly look like; we only need a bit more information to have it nailed down precisely, one measurement for each constant.

Of course, in the real world, measurements come with errors so we need to take more than two. But we can get a lot of information from our differential equation telling us what sort of relationships to look for.

Example 3.13. Suppose $f(x) = ax^2 + bx + c$ is a polynomial satisfying some differential equation, and we have $f(0) = 0, f'(0) = 1, f''(0) = 2$. What can we say about $f(x)$?

We see that $f(0) = c = 0$, $f'(x) = 2ax + b$ so $f'(0) = b = 1$, and $f''(x) = 2a$ so $f''(0) = 2a = 2$. Thus $a = b = 1$ and $c = 0$, so $f(x) = x^2 + x$.

Example 3.14. Suppose $g(x) = ax^2 + bx + c$ is a polynomial satisfying some differential equation, with $g(1) = 2, g'(2) = 3, g''(3) = 4$. What can we say about g?

We have $g(1) = a + b + c$. $g'(x) = 2ax + b$ so $g'(2) = 4a + b = 3$, and $g''(x) = 2a$ so $g''(3) = 2a = 4$. Thus we have $a = 2$. Going back to g' we see that $8 + b = 3$ so $b = -5$. Then plugging into g we have $2 - 5 + c = 2$ so $c = 5$. Thus $g(x) = x^2 - 5x + 5$.

3.3 Euler's Method

We cannot develop a general method of *exactly* solving even simple differential equations in this course, since this requires integrals. But we can come up with approximate solutions. And sometimes approximate solutions are the best that anyone can do!

Example 3.15. Let's consider about the simplest possible non-trivial differential equation: $f'(x) = f(x)$. And let's add in the information that $f'(0) = 1$. What can we say about the values of this function?

Let's start by approximating $f(1)$. From our differential equation we know that $f'(0) = f(0) = 1$, so by linear approximation we have

$$f(1) \approx f'(0)(1-0) + f(0) = 1 + 1 = 2.$$

But of course this isn't an exact answer. Can we be more precise?

Recall our approximations get less and less accurate as we get farther away from our base point, because the derivative keeps changing. We can improve our accuracy by stopping halfway through to correct our estimate of the rate of change.

$$\begin{aligned} f(.5) &\approx f'(0)(.5-0) + f(0) = 1 \cdot .5 + 1 = 1.5 \\ f(1) &\approx f'(.5)(1-.5) + f(.5) \approx 1.5(.5) + 1.5 = 2.25. \end{aligned}$$

We can always get more precision by using more steps.

$$\begin{aligned} f(1/4) &\approx f'(0)(1/4-0) + f(0) = 1(1/4) + 1 = 5/4 \\ f(1/2) &\approx f'(1/4)(1/2 - .1/4) + f(1/4) \approx 5/4(1/4) + 5/4 = 25/16 \\ f(3/4) &\approx f'(1/2)(3/4 - 1/2) + f(1/2) \approx \frac{25}{16}(1/4) + \frac{25}{16} = \frac{125}{64} \\ f(1) &\approx f'(3/4)(1-3/4) + f(3/4) \approx \frac{125}{64} \cdot \frac{1}{4} + \frac{125}{64} = \frac{625}{256} \approx 2.44. \end{aligned}$$

We will see in Section 3.4 that the exact solution to this problem is $e \approx 2.71828$.

(We will make a note to recall later that with one step, we had $(1+1)^1$; with two steps, we had $(3/2)^2$; and with four steps we had $(5/4)^4$. We can see that in general, with n steps we will have $((n+1)/n)^n$ as our approximation).

This approach to approximating solutions to a differential equation is known as *Euler's Method*:

1. Pick a step size h.
2. Start with a base point whose value is known: $f(x_0) = y_0$.
3. Use the differential equation to compute $f'(x_0)$.
4. Use a linear approximation to approximate $f(x_0 + h)$.

5. Take this as your new base point, compute $f'(x_0+h)$, and then approximate $f(x_0+2h)$.

6. Repeat until you have approximated your desired output.

Example 3.16. Suppose $f'(t) = f(t) - f(t)^2/2$, and $f(0) = 1$. Let us approximate $f(3)$ using 3 steps, for a step size of 1.

$$f(1) \approx f'(0)(1-0) + f(0) = (1 - 1^2/2)(1) + 1 = 3/2.$$
$$f(2) \approx f'(1)(2-1) + f(1) \approx \left(\frac{3}{2} - \frac{\left(\frac{3}{2}\right)^2}{2}\right)(1) + \frac{3}{2} = \frac{3}{8} + \frac{3}{2} = \frac{15}{8}.$$
$$f(3) \approx f'(2)(3-2) + f(2) \approx \left(\frac{15}{8} - \frac{\left(\frac{15}{8}\right)^2}{2}\right)(1) + \frac{15}{8} = \frac{15}{128} + \frac{15}{8} = \frac{255}{128}.$$

Thus we estimate $f(3) \approx 1.99$.

Example 3.17. Suppose $f'(x) = x - f(x)$ and $f(1) = 3$. Let's approximate $f(2)$ with a step size of $1/4$. We have

$$f(5/4) \approx f'(1)(1/4) + f(1) = (1-3)(1/4) + 3 = \frac{5}{2}$$
$$f(3/2) \approx f'(5/4)(1/4) + f(5/4) \approx \left(\frac{5}{4} - \frac{5}{2}\right)\frac{1}{4} + \frac{5}{2} = \frac{35}{16}$$
$$f(7/4) \approx f'(3/2)(1/4) + f(3/2) \approx \left(\frac{3}{2} - \frac{35}{16}\right)\frac{1}{4} + \frac{35}{16} = \frac{129}{64}$$
$$f(2) \approx f'(7/4)(1/4) + f(7/4) \approx \left(\frac{7}{4} - \frac{129}{64}\right)\frac{1}{4} + \frac{129}{64} = \frac{499}{256}.$$

Thus we estimate $f(2) \approx \frac{499}{256} \approx 1.95$.

We can also use this same method to solve more complicated systems of equations.

Example 3.18. Let's return to our Hooke's Law equation $x''(t) = -x(t)$, and suppose we know that $x(0) = 3$ and $x'(0) = -1$. What is $x(3)$?

We know at each step that $x(t+h) \approx x(t) + x'(t)h$, but this alone doesn't help us since we don't have a formula for $x'(t)$. But we also know that $x'(t+h) \approx x'(t) + x''(t)h$, and these two facts together let us use an Euler's Method approach.

We can approximate

$$\begin{aligned}
x(1) &\approx x(0) + x'(0)(1-0) = 3 - 1 \cdot 1 = 2 \\
x'(1) &\approx x'(0) + x''(0)(1-0) = (-1) + (-x(0)) = -4 \\
x(2) &\approx x(1) + x'(1)(2-1) \approx 2 - 4 = -2 \\
x'(2) &\approx x'(1) + x''(1)(2-1) \approx -4 + (-2) = -6 \\
x(3) &\approx x(2) + x'(2)(3-2) \approx -2 - 6 = -8.
\end{aligned}$$

3.4 Exponential Growth

Finally let's return to our first and simplest differential equation $y' = y$. Since this equation shows up so often—and is also a nice derivative rule on its own—it would be nice to find a way of computing this function explicitly.

Let's take the initial condition $y(0) = 1$. Then Euler's method gives us that

$$y(h) \approx y(0) + y'(0)(h-0) = y(0) + y(0)h = y(0)(1+h).$$

More generally, whenever I know some value $y(a)$ then I get

$$y(a+h) \approx y(a) + y'(a)(a+h-a) = y(a) + hy(a) = (1+h)y(a).$$

Now suppose we want to approximate $y(1)$ in n steps. We know we will write

$$\begin{aligned}
y(1/n) &\approx y(0) + y'(0)(1/n - 0) = y(0)(1+1/n) \\
y(2/n) &\approx y(1/n) + y'(1/n)(1/n) = y(1/n)(1+1/n) = y(0)(1+1/n)^2 \\
y(3/n) &\approx y(0)(1+1/n)^3 \\
&\vdots \\
y(1) &\approx y(0)(1+1/n)^n = (1+1/n)^n.
\end{aligned}$$

We know our approximation improves as the number of steps increases, so we might want to ask what happens as the number of steps tends to infinity. It's not too hard to see that $\lim_{n\to+\infty}(1+1/n)^n$ is finite; thus it is a specific, constant number, and we can give it a name. Historically it was named by and in honor of the mathematician Leonhard Euler, and thus this number is known as Euler's constant, or just as e. We can approximate its value by computing according to Euler's method for sufficiently large n.

Definition 3.19. *Euler's constant* is the number $e = \lim_{n\to+\infty}(1+1/n)^n \approx 2.71828$.

Moreover, we can find the values of $y(x)$ for any x! We can approximate $x \approx k/n$ for some k; this approximation may be inexact, but as n gets larger, the approximation will be more precise. Then we have

$$y(x) \approx y(0)(1+1/n)^k = 1 \cdot ((1+1/n)^n)^{k/n} \approx (1+1/n)^n)^x$$
$$y(x) = \lim_{n \to +\infty} ((1+1/n)^n)^x = e^x.$$

Thus if $y = y'$ and $y(0) = 1$, we know that $y(x) = e^x$; so e^x is a solution to this initial value problem.

In fact, every solution to $y = y'$ has must have this form.

Proposition 3.20. *Consider the differential equation $y' = y$. Then it must be the case that $y(x) = Ce^x$ for some constant C.*

Proof. Suppose $f(x)$ satisfies the differential equation, so that $f'(x) = f(x)$. Then we can consider the derivative of $\frac{f(x)}{e^x}$. By the product rule we have

$$\frac{d}{x}\frac{f(x)}{e^x} = \frac{f'(x)e^x - f(x)\frac{d}{dx}e^x}{(e^x)^2} = \frac{f(x)e^x - f(x)e^x}{e^{2x}} = 0.$$

Thus the function $\frac{f(x)}{e^x}$ is a constant, and so we have $\frac{f(x)}{e^x} = C$ or $f(x) = Ce^x$.

□

Example 3.21 (Compound Interest)**.** Leonhard Euler wasn't actually the first person to discover the constant e; it was earlier studied by Jakob Bernoulli in the context of compound interest payments.

Suppose you invest \$100 in a bank account paying 3% interest a year. Then after t years you will have $100 \cdot (1.03)^t$ dollars in the bank account. It's easy to compute how much money you'll have after t years. For instance, after three years you will have \$109 and after 20 years you will have \$180.

Often interest is "compounded" more often, meaning that you get some fraction of it every few months. Interest that is compounded quarterly—four times a year—pays you .75% of your current balance four times a year, so after t years you will have $100 \cdot (1.0075)^{4t}$ dollars. After three years you will still have \$109, and after 20 years you will have \$182. Note that your money has increased—slightly.

We can compound more often; in general, if your interest rate is r and you compound n times a year, then your total money after t years will be

$$M = M_0(1 + \frac{r}{n})^{nt},$$

where M_0 is the amount of money you started with.

In the real economy, transactions are constantly happening and the economy is (usually) constantly growing. Jacob Bernoulli asked what would happen if your interest *compounded continuously*—that is, what happens in the limit, as n goes to $+\infty$.

$$\begin{aligned} M(t) &= \lim_{n\to+\infty} M_0\left(1+\frac{r}{n}\right)^{nt} \\ &= M_0 \lim_{n\to+\infty}\left(\left(1+\frac{r}{n}\right)^{n/r}\right) \\ &= M_0\left(\lim_{n\to+\infty}\left(1+\frac{r}{n}\right)^{n/r}\right)^{rt} \\ &= M_0 e^{rt}. \end{aligned}$$

And now you see (one reason) why e is considered a "natural" base for the exponential map.

With this, we can ask how long it will take for us to have \$200 if our interest is compounded continuously. We have

$$\begin{aligned} 200 &= 100e^{.03t} \\ 2 &= e^{.03t} \\ \ln(2) &= .03t \\ 23 &= t \end{aligned}$$

so it will take about 23 years to double our money.

Remark 3.22. We know that $\ln(2) = .693\cdots \approx .7$. This gives us the useful rule of thumb that if your interest rate is r, it will take about $70/r$ years to double your investment.

We can also use the differential equation $y = y'$ to tell us facts about the function e^x.

Definition 3.23. We define the function exp by $\exp(x) = e^x$.

Then exp is a solution to the differential equation $y = y'$. Thus we know that $\frac{d}{dx}\exp(x) = \exp(x)$, or in other words $\frac{d}{dx}e^x = e^x$.

This gives us another differentiation rule! We can combine this with the things we already know about differentiation.

Example 3.24. If $f(x) = e^{kx}$ then $f'(x) = e^{kx}\cdot(kx)' = ke^{kx}$. Thus e^{kx} is a solution to the differential equation $y = ky'$.

Example 3.25. If $f(x) = e^{\sin(x)}$ then $f'(x) = e^{\sin(x)}\cdot\cos(x)$.

If $g(x) = \frac{e^x+e^{-x}}{x}$ then

$$g'(x) = \frac{(e^x - e^{-x})x - (e^x + e^{-x})}{x^2}.$$

3.5 Relationships

We defined a function as a rule, that takes some input and gives some output. Usually we give you the rule explicitly, as when we say $y = x^2 - 1$. But sometimes you only know facts about the rule, such as $y^2 + x^2 = 1$ (which describes the unit circle). Sometimes these facts will describe one function uniquely, and sometimes they won't. (This comes up a lot in solving actual problems in physics and economics and other fields).

Regardless of where we get an equation like this, we know that both sides are equal, so the derivatives of both sides are equal. So using the chain rule and thinking of y as a function of x, we can simply take derivatives of both sides, and then do some algebra to find y'.

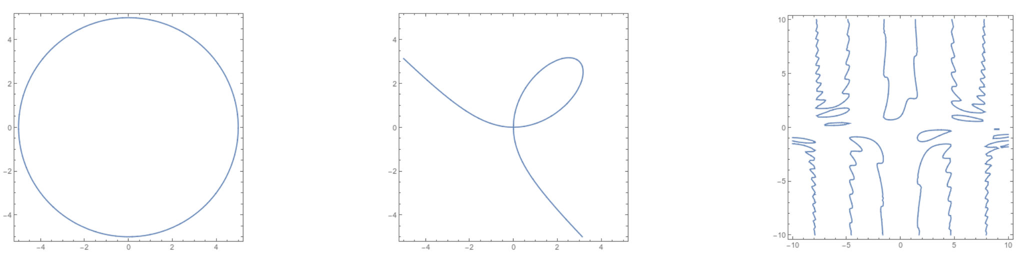

Figure 3.1: Left: The circle $x^2 + y^2 = 25$. Center: the folium of Descartes $x^3 + y^3 = 6xy$. Right: $y\cos(x) = 1 + \sin(xy)$

If we want to find tangent lines for these curves, we can use implicit differentiation. Essentially, we take the derivative of both sides of the equation, treating y as a function of x and applying the chain rule.

Example 3.26. • If $x^2 + y^2 = 25$, then

$$\begin{aligned}\frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}(25) \\ 2x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-x}{y}.\end{aligned}$$

Thus at the point, say, $(3, 4)$ (check that this is on the circle!), we have that $\frac{dy}{dx}(3, 4) = \frac{-3}{4} = -3/4$. Thus the equation of the line tangent to the circle at $(3, 4)$ is $y - 4 = -\frac{3}{4}(x - 3)$.

- If $x^3 + y^3 = 6xy$, then

$$\begin{aligned}
\frac{d}{dx}\left(x^3 + y^3\right) &= \frac{d}{dx}(6xy) \\
3x^2 + 3y^2\frac{dy}{dx} &= 6\left(y + x\frac{dy}{dx}\right) \\
(3y^2 - 6x)\frac{dy}{dx} &= 6y - 3x^2 \\
\frac{dy}{dx} &= \frac{6y - 3x^2}{3y^2 - 6x}.
\end{aligned}$$

At the point $(0,0)$ this doesn't actually give us a useful answer; if you look at the picture in Figure 3.1, you see that there's not a clear tangent line there since the curve crosses itself.

In contrast, at the point $(3,3)$ we have that

$$\frac{dy}{dx} = \frac{18 - 27}{27 - 18} = -1$$

and the equation of the tangent line is $y - 3 = -(x - 3)$.

- If $y\cos(x) = 1 + \sin(xy)$, then

$$\begin{aligned}
\frac{d}{dx}(y\cos(x)) &= \frac{d}{dx}(1 + \sin(xy)) \\
\frac{dy}{dx}\cos(x) - y\sin(x) &= \cos(xy)\left(y + x\frac{dy}{dx}\right) \\
\frac{dy}{dx}(\cos(x) - x\cos(xy)) &= y\cos(xy) + y\sin(x) \\
\frac{dy}{dx} &= \frac{y\cos(xy) + y\sin(x)}{\cos(x) - x\cos(xy)}.
\end{aligned}$$

- If $\sqrt{xy} = 1 + x^2 y$, then

$$\begin{aligned}
\frac{d}{dx}\sqrt{xy} &= \frac{d}{dx}\left(1 + x^2 y\right) \\
\frac{1}{2}(xy)^{-1/2}\left(y + x\frac{dy}{dx}\right) &= 2xy + x^2\frac{dy}{dx} \\
\frac{dy}{dx}\left(x^2 - \frac{1}{2}x(xy)^{-1/2}\right) &= \frac{1}{2}(xy)^{-1/2}y - 2xy \\
\frac{dy}{dx} &= \frac{\frac{1}{2}(xy)^{-1/2}y - 2xy}{x^2 - \frac{1}{2}x(xy)^{-1/2}}.
\end{aligned}$$

Example 3.27. We can also compute second derivatives implicitly. If $9x^2 + y^2 = 9$ then we have

$$\begin{aligned}
18x + 2y\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= -\frac{9x}{y} \\
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(-\frac{9x}{y}\right) \\
&= -\frac{9y - 9x\frac{dy}{dx}}{y^2} \\
&= -\frac{9y - 9x(-\frac{9x}{y})}{y^2} \\
&= -\frac{9y + \frac{81x^2}{y}}{y^2}
\end{aligned}$$

We see that at the point $(0, 3)$ we have $y' = 0$ and $y'' = -3$. At the point $(\sqrt{5}/3, 2)$, then $y' = -\frac{3\sqrt{5}}{2}$ and $y'' = -\frac{18+\frac{45}{2}}{4}$.

Example 3.28. Find y'' if $x^6 + \sqrt[3]{y} = 1$. Then find the first and second derivatives at the point $(0, 1)$.

$$\begin{aligned}
6x^5 + \frac{1}{3}y^{-2/3}y' &= 0 \\
-18x^5y^{2/3} &= y' \\
-18(5x^4y^{2/3} + \frac{2}{3}x^5y^{-1/3}y') &= y'' \\
-18(5x^4y^{2/3} + \frac{2}{3}x^5y^{-1/3}(-18x^5y^{2/3})) &= y''
\end{aligned}$$

Thus at $(0, 1)$, we have $y' = 0$ and $y'' = 0$. So the tangent line to the curve is horizontal at the point $(0, 1)$.

We can also use implicit differentiation on relationships that apply to functions.

Example 3.29. Suppose we have some function f such that $8f(x) + x^2(f(x))^3 = 24$, and we want to find a linear approximation of f near $f(4) = 1$. (Say we've measured this experimentally and now want to understand or compute with the function). Then we have

$$\begin{aligned}\frac{d}{dx}\left(8f(x) + x^2(f(x))^3\right) &= \frac{d}{dx}24 \\ 8f'(x) + 2x(f(x))^3 + 3x^2(f(x))^2 f'(x) &= 0 \\ 8f'(4) + 2 \cdot 4 \cdot 1^3 + 3 \cdot 4^2 \cdot 1^2 f'(4) &= 0 \\ 8f'(4) + 8 + 48f'(4) &= 0\end{aligned}$$

and thus $f'(4) = -1/7$.

This leaves us with a question, though. We know $f(1)$; can we figure out the value of f at other points?

We have a derivative, so we can again compute a linear approximation. We get

$$f(x) \approx f'(4)(x-4) + f(4) = \frac{-1}{7}(x-4) + 1.$$

Thus we compute

$$f(5) \approx \frac{-1}{7}(5-4) + 1 = 1 + \frac{-1}{7} = \frac{6}{7} \approx .857.$$

Checking Mathematica, we see that the actual solution is .879. So we were pretty close.

Sometimes we have word problems that require us to translate verbal information into equations, and then solve the problem.

Checklist of steps for solving word problems:

1. Draw a picture.
2. Think about what you expect the answer to look like. What is physically plausible?
3. Create notation, choose variable names, and label your picture.
   (a) Write down all the information you were given in the problem.
   (b) Write down the question in your notation.
4. Write down equations that relate the variables you have.
5. Abstractly: "solve the problem." Concretely differentiate your equation.
6. Plug in values and read off the answer.

7. Do a sanity check. Does you answer make sense? Are you running at hundreds of miles an hour, or driving a car twenty gallons per mile to the east?

Example 3.30. Suppose one car drives north at 40 mph, and an hour later another starts driving west from the same place at 60 mph. After a second hour, how quicly is the distance between them increasing?

Write a for the distance the first car has traveled, and b for the distance the second car has traveled. We have that $a = 80, b = 60, a' = 40, b' = 60$. If the distance between the cars is d then after two hours, $d = 100$, and we have

$$\begin{aligned} d^2 &= a^2 + b^2 \\ 2dd' &= 2aa' + 2bb' \\ 2 \cdot 100 \cdot d' &= 2 \cdot 80 \cdot 40 + 2 \cdot 60 \cdot 60 \\ d' &= \frac{3200 + 3600}{100} = 68, \end{aligned}$$

so the distance between the cars is increasing at 68 mph. This seems reasonable because the cars are traveling at 40 mph and 60 mph.

Example 3.31. A twenty foot ladder rests against a wall. The bit on the wall is sliding down at 1 foot per second. How quickly is the bottom end sliding out when the top is 12 feet from the ground?

Let h be the height of the ladder on the wall, and b be the distance of the foot of the ladder from the wall. Then $h = 12, h' = -1$, and $b = \sqrt{400 - 144} = 16$. We have

$$\begin{aligned} h^2 + b^2 &= 400 \\ 2hh' + 2bb' &= 0 \\ 2 \cdot 12 \cdot (-1) + 2 \cdot 16 \cdot b' &= 0 \\ b' = \frac{24}{32} &= 3/4 \end{aligned}$$

so the foot of the ladder is sliding away from the wall at 3/4 ft/s. Again, the direction of the sliding is correct (away from the wall), and the number seems plausible.

Example 3.32. A spherical balloon is inflating at 12 cm^3 per second. How quickly is the radius increasing when the radius is 3 cm?

A sphere has volume $V = \frac{4}{3}\pi r^3$. We have $V' = 12$ and $r = 3$. We compute

$$\begin{aligned} V' &= 4\pi r^2 r' \\ 12 &= 4\pi(3)^2 r' \\ r' &= \frac{1}{3\pi} \end{aligned}$$

So the radius is increasing by $1/3\pi$ cm per second.

Example 3.33. A rectangle is getting longer by one inch per second and wider by two inches per second. When the rectangle is 5 inches long and 7 inches wide, how quickly is the area increasing?

We have $l = 5, w = 7, l' = 1, w' = 2$, and $A = lw$. Taking a derivative gives us $A' = lw' + wl' = 5 \cdot 2 + 7 \cdot 1 = 17$ square inches per second.

Example 3.34. An inverted conical water tank with radius 2m and height 4m is being filled with water at a rate of $2\text{m}^3/\text{min}$. How fast is the water rising when the water is 3 m tall?

Let h be the current height of the water, r the current radius, and V the current volume of water. We know that $h = 3$, and by similar triangles we see that $\frac{h}{r} = \frac{4}{2}$ and thus $r = h/2$. We know that $V' = 2$, and the volume formula for a cone gives us $V = \frac{1}{3}\pi r^2 h$. We compute

$$\begin{aligned} V &= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{3}\pi\frac{h^3}{4} \\ V' &= \frac{\pi}{4}h^2 h' \\ 2 &= \frac{\pi}{4}3^2 h' \\ \frac{8}{9\pi} &= h', \end{aligned}$$

so the water level is rising at $\frac{8}{9\pi}$ meters per minute.

Example 3.35. A street light is mounted at the top of a 15-foot-tall pole. A six-foot-tall man walks straight away from the pole at 5 feet per second. How fast is the tip of his shadow moving when he is forty feet from the pole?

Let d be the distance of the man from the pole, and L be the distance from the pole to the tip of his shadow. We have $d' = 5$ and we set up a similar triangles equation.

$$\begin{aligned} \frac{15}{L} &= \frac{6}{L-d} & 6L &= 15L - 15d \\ 9L &= 15d & d &= \frac{3}{5}L \\ d' &= \frac{3}{5}L' & 5 &= \frac{3}{5}L' \end{aligned}$$

and thus the tip of his shadow is moving at $\frac{25}{3}$ feet per second.

Example 3.36. A lighthouse is located three kilometers away from the nearest point P on shore, and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline 1 kilometer from P?

Let's say the angle of the light away from P is θ, and the distance from P is d. Then we have $d = 1$ and $\theta' = 8\pi$ (in radians per minute). We also have the relationship that $\tan\theta = \frac{d}{3}$.

Taking the derivative gives us $\sec^2(\theta) \cdot \theta' = d'/3$. We need to work out $\sec^2(\theta)$, but looking at our triangle we see that the adjacent side is length 3 and the hypotenuse is length $\sqrt{10}$ (by the Pythagorean theorem), so we have $\sec^2(\theta) = (\sqrt{10}/3)^2 = 10/9$.

Thus we have $d' = 3\sec^2(\theta) \cdot 8\pi = \frac{80\pi}{3}$ kilometers per second.

Example 3.37. A kite is flying 100 feet over the ground, moving horizontally at 8 ft/s. At what rate is the angle between the string and the ground decreasing when 200ft of string is let out?

Call the distance between the kite-holder and the kite d and the angle between the string and the ground θ. When the length of string is 200 then $d = \sqrt{200^2 - 100^2} = 100\sqrt{3}$. We have that $d' = 8$ (since the angle is decreasing, the kite must be getting farther away). And finally we have the relationship $\tan\theta = \frac{100}{d}$ by the definition of tan in terms of triangles. Then we have

$$\begin{aligned} \tan\theta &= 100d^{-1} \\ \sec^2(\theta)\theta' &= -100d^{-2}d' \\ \theta' &= \frac{-100 \cdot 8\cos^2(\theta)}{d^2}. \end{aligned}$$

We see that $\cos(\theta) = \frac{100\sqrt{3}}{200} = \sqrt{3}2$, so we have

$$\theta' = \frac{-100 \cdot 8 \cdot 3/4}{(100\sqrt{3})^2} = -\frac{8}{100 \cdot 4} = \frac{-1}{50}.$$

So the angle between the string and the ground is decreasing at a rate of $1/50$ per second. (Note: radians are unitless!)