4 Undoing: Inverses and Solving Equations

4.1 Inverse Functions

Remember we started out by saying that a function is a process: it takes an input and returns an output. Sometimes we want to undo this process. This is in fact a natural question; "What do I have to do if I want to get X" is a pretty common thought process. So our goal is: given a function f, given $f(x)$, can we find x?

Definition 4.1. If f is a function and $(g \circ f)(x) = x$ for every x in the domain of f, then we say g is an *inverse* of f.

Example 4.2.

- If $f(x) = x$ then $g(y) = y$ is an inverse to f.
- If $f(x) = 5x + 3$ then $g(y) = (y - 3)/5$ is an inverse to f.
- If $f(x) = x^3$ then $g(y) = \sqrt[3]{y}$ is an inverse to f.

Remark 4.3. A given function f has at most one inverse—if f has an inverse at all, then that means "for any y, find the x where $f(x) = y$" is a well-defined rule.

If g is an inverse to f, then the domain of g is the image of f and the domain of f is the image of g.

Computing $f^{-1}(y)$ is the same as solving the equation $f(x) = y$.

Unfortunately, we can't always find these inverses. For instance, if you know that $x^2 = 9$, you don't know for sure what x is: it could be 3 or -3. Similarly, if you know $\sin(x) = 0$, then x could be $n\pi$ for any integer n. The fundamental problem here is that there are some outputs that are generated by more than one input.

Definition 4.4. A function f is *1-1* or *one-to-one* (or *injective*) if, whenever $f(a) = f(b)$, we know that $a = b$.

Example 4.5. Functions which are 1-1:

- $f(x) = x$. If $f(a) = f(b)$ then $a = b$ by definition.
- $f(x) = x^3$. If $f(a) = f(b)$ then $a^3 = b^3$, and then $(a/b)^3 = 1$ so $a/b = 1$ and $a = b$.
- $f(x) = \sqrt{x}$. If $f(a) = f(b)$ then $\sqrt{a} = \sqrt{b}$ so $|a| = |b|$. But $a, b \geq 0$ since they're in the domain of f, and thus $a = b$.

Functions which are not 1-1:

- $f(x) = x^2$, since $f(-1) = f(1)$.
- $f(x) = |x|$, since $f(-2) = f(2)$.
- $\sin(x)$, since $\sin(0) = \sin(\pi)$.
- $f(x) = 3$, since $f(a) = f(b) = 3$ for any real numbers a and b.

However, we can often force a function to be one-to-one by restricting its domain.

Example 4.6.
- The function $f(x) = x^2$ on the domain $[0, +\infty)$ is 1-1. If $f(a) = f(b)$ then $a^2 = b^2$ so $a = \pm b$. But both $a, b \geq 0$ so $a = b$.
- The function $\sin(x)$ is 1-1 on the domain $[-\pi/2, \pi/2]$. If we look at the unit circle, we see that as x varies from $-\pi/2$ to $\pi/2$, the y coordinate on the unit circle is always increasing, and so never repeats itself.

This might lead us to think graphically about what the idea of 1-1-ness means:

Proposition 4.7 (Horizontal Line Test)**.** *A function f is 1-1 if and only if any horizontal line will intersect its graph in at most one point.*

It's reasonably clear that every function with an inverse must be one-to-one, since otherwise there's not a unique answer to the inverse question. Less obvious is that being 1-1 is enough to be invertible.

Proposition 4.8. *If f is a 1-1 function with domain A and image B, then there is a function f^{-1} with domain B and image A which is an inverse to f.*

We can find this inverse by writing the equation $y = f(x)$ and solving for x as a function of y. Finding an inverse for f is also a good way to prove that f is one-to-one.

Example 4.9. Let $f(x) = x^4$ with domain $(-\infty, 0]$. Then we have $y = x^4 \Rightarrow x = \pm\sqrt[4]{y}$. But we know that $x < 0$ so $x = -\sqrt[4]{y}$, and thus $g(y) = -\sqrt[y]{y}$ is an inverse for f.

Graphically, the graph of f^{-1} looks like the graph of f flipped acroos the line $y = x$, which makes sense, since a point (x, y) on the graph of f should correspond to a point (y, x) on the graph of f^{-1}. In fact, the Horizontal Line Test mentioned earlier is basically the Vertical Line Test applied to the inverse function.

Example 4.10. Take $f(x) = x^3 - x$. This function is clearly not one-to-one, since $f(1) = f(0) = f(-1) = 0$. But we can split it up into intervals where it is one-to-one. Looking at the graph, it seems natural to split it up at the critical points. And this suggests we should use calculus to study our inverse function problem.

4.1.1 Calculus of inverse functions

Now that we understand inverse functions as functions, we'd like to see what calculus can tell us about them.

Proposition 4.11. *If f is one-to-one and continuous at a, then f^{-1} is continuous at $f(a)$.*

If f is one-to-one and continuous, then f^{-1} is continuous.

We'd really like to know about the derivatives of inverse functions. We can work out what they are with some quick sketched arguments, and then can prove the answer rigorously once we know what we're looking for.

First, the argument by "it looks nice in the notation": we can rephrase this theorem as saying that

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Second, if we already know that both functions are differentiable, we can use implicit differentiation:

$$\begin{aligned} f^{-1}(f(x)) &= x \\ (f^{-1})'(f(x)) \cdot f'(x) &= 1 \\ (f^{-1})'(f(x)) &= \frac{1}{f'(x)}. \end{aligned}$$

Writing $x = f^{-1}(a)$, or equivalently $a = f(x)$, gives our statement.

Theorem 4.12 (Inverse Function Theorem)**.** *If f is a one-to-one differentiable function, and $f'(f^{-1}(a)) \neq 0$, then $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$.*

Proof. Set $y = f^{-1}(x)$ and $b = f^{-1}(a)$. Then

$$\begin{aligned} (f^{-1})'(a) &= \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a} \\ &= \lim_{y \to b} \frac{y - b}{f(y) - f(b)} \\ &= \lim_{y \to b} \frac{1}{\frac{f(y) - f(b)}{y - b}} \\ &= \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a))}. \end{aligned}$$

□

Example 4.13. Let $f(x) = x^n$ on $[0, +\infty)$; then $f^{-1}(x) = \sqrt[n]{x}$. Our formula gives

$$\begin{aligned}(f^{-1})'(a) &= \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(\sqrt[n]{(a)})} \\ &= \frac{1}{n(\sqrt[n]{(a)})^{n-1}} = \frac{1}{na^{(n-1)/n}} = \frac{1}{n}a^{(1-n)/n} = \frac{1}{n}a^{\frac{1}{n}-1}.\end{aligned}$$

Though at first this didn't look like our original answer, it is the same as the formula we had before.

Example 4.14. Let $f(x) = \sqrt[3]{5x^2+7}$. What is $(f^{-1})'(3)$?

Well, we have $(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))}$. We know that $f'(x) = \frac{1}{3}(5x^2+7)^{-2/3} \cdot 10x$, and we can work out that $f(2) = \sqrt[3]{20+7} = 3$ (by plugging in small integers until one works). Thus $f^{-1}(3) = 2$, and so we have

$$(f^{-1})'(3) = \frac{1}{\frac{1}{3}(27)^{-2/3} \cdot 20} = \frac{3 \cdot 9}{20} = \frac{27}{20}.$$

4.2 Intermediate Value Theorem

When we compute an inverse function, we are implicitly solving an equation. For a given function f, computing $f^{-1}(y)$ is the same as solving the equation $f(x) = y$ for the variable x. Now, if we can explicitly solve our equation, this is perfectly reasonable. But while I can write down the formula $\sin^{-1}(1/3)$, I can't actually tell you an x such that $\sin(x) = 1/3$. So a reasonable question to ask is how we know equations like this even have solutions.

The idea of continuity comes to the rescue. In section 1.3 we defined a continuous function as one where $\lim_{x\to a} f(x) = f(a)$. But a common informal definition is that a continuous function is one whose we can draw without lifting our pencil from the paper. Once we make this precise, this is another way to think about continuous functions. And we make it precise via the Intermediate Value Theorem

Theorem 4.15 (Intermediate Value Theorem). *Suppose f is continuous (and defined!) on the closed interval $[a, b]$ and y is any number between $f(a)$ and $f(b)$. Then there is a c in (a, b) with $f(c) = y$.*

Example 4.16. Suppose $f(x)$ is a continuous function with $f(0) = 3, f(2) = 7$. Then by the Intermediate Value Theorem there is a number c in $(0, 2)$ with $f(c) = 5$.

Example 4.17. Let $g(x) = x^3 - x + 1$. Use the Intermediate Value Theorem to show that there is a number c such that $g(c) = 4$.

To use the intermediate value theorem, we need to check that our function is continuous, and then find one input whose output is less than 4, and another whose output is greater than 4. g is a polynomial and thus continuous. Testing a few values, we see $g(0) = 1, g(1) = 1, g(2) = 7$. Since $g(1) = 1 < 4 < 7 = g(2)$, by the Intermediate Value Theorem ther is a c in $(1, 2)$ with $g(c) = 4$.

Example 4.18. Show that there is a θ in $(0, \pi/2)$ such that $\sin(\theta) = 1/3$.

We know that sin is a continuous function, and that $\sin(0) = 0$ and $\sin(\pi/2) = 1$. Since $0 < 1/3 < 1$, by the Intermediate Value Theorem there is a θ in $(0, \pi/2)$ such that $\sin(\theta) = 1/3$.

Remark 4.19. The converse of this theorem is not true. It is possible to have a function that satisfies the conclusions of the Intermediate Value Theorem, but is not continuous; these functions are called Darboux Functions.

For example, let $f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$. Then f satisfies the conclusion of the intermediate value theorem: it's continuous except at zero, so the theorem works on any interval that doesn't contain zero. Any interval containing zero contains every value in $[-1, 1]$, so if $a < 0 < b$ and y is between $f(a)$ and $f(b)$, then $-1 \leq y \leq 1$ and so there is a c in (a, b) such that $f(c) = y$. Thus f is Darboux.

Historically, the main reason we didn't take this as the definition of continuous, instead of the limit definition that we actually use, is that no one looks at that function and thinks it should count.

In this section we have asked a lot of questions about solving equations. But whenever we have an equation of the form $f(x) = b$, we can just subtract b from both sides and get $f(x) - b = 0$. We can now define a new function $g(x) = f(x) - b$, and our equation is $g(x) = 0$. So if we understand solving equations with zero on one side really well, we secretly understand all the possible equations.

In practice, this is how we actually develop tools to solve equations. We always assume there's a 0 on one side, because if there isn't that problem is easy to fix. You're already familiar with this from solving quadratic equations. If I ask you to solve the equation $x^2 = 3x + 4$ the first thing you probably do is move all the terms to one side so you can factor.

Because we want to look at this special form of equation, we have some specific terms for dealing with the situation.

Definition 4.20. We say a function f has a *root* at c, or that c is a root of f, if $f(c) = 0$.

We often use the Intermediate Value Theorem to show that a continuous function must have a root.

Example 4.21. Show that $f(x) = x - \sqrt{x} - 1$ has a root.

f is continuous since it is composed of polynomials and roots. Plugging in values, we see that $f(0) = -1, f(1) = -1, f(2) = 1 - \sqrt{2}, f(3) = 2 - \sqrt{3}, f(4) = 1$. We see that $f(0) = -1 < 0 < 1 = f(4)$, so by the Intermediate Value Theorem there is a c between 0 and 4 such that $f(c) = 0$.

Poll Question 4.2.1. Does $x^2 + 2$ have a root?

Does $x^2 - 2$ have a root?

Why are these two situations different?

Poll Question 4.2.2. Does $\cos x = x$ anywhere between 0 and 1? Between 1 and 2?

Example 4.22. Use the Intermediate Value Theorem to show that $f(x) = \tan(x) + \sin(x) - 1$ has a root between 0 and $\pi/4$.

f is continuous since it is made of trig functions (we need to check that every point between 0 and $\pi/4$ is in the domain, but this is true). $f(0) = 0 + 0 - 1 = -1$ and $f(\pi/4) = 1 + \sqrt{2}/2 - 1 = \sqrt{2}/2$. Since $-1 < 0 < \sqrt{2}/2$, by the Intermediate Value Theorem f has a root in $(0, \pi/4)$.

Example 4.23. Does $g(x) = \tan(x)$ have a root between $\pi/4$ and $3\pi/4$?

g is continuous where defined, and $g(\pi/4) = 1, g(3\pi/4) = -1$. However, g is not defined on the closed interval $[\pi/4, 3\pi/4]$ since $\tan(\pi/2)$ is not defined. Thus the Intermediate Value Theorem does not apply. Graphing the function, we can see it has no roots on this interval.

Example 4.24. Use the Intermediate Value Theorem to show that $h(x) = x^4 - 3$ has two *distinct* roots.

As usual, we begin by plugging in some values. We see that $h(-2) = 13, h(-1) = -2, h(0) = -3, h(1) = -2, h(2) = 13$. Thus since $h(-2) = 13 > 0 > -3 = h(0)$, by the Intermediate Value Theorem h has a root between -2 and 0. Similarly, since $h(0) = -3 < 0 < 13 = h(2)$, by the Intermediate Value Theorem h has a root between 0 and 2.

Since no number can simultaneously be in $(-2, 0)$ and $(0, 2)$, these two roots cannot be the same. Thus h has two distinct roots.

Example 4.25. Use the Intermediate Value Theorem to show that $f(x) = x^3 - 3x + 1$ has *three* distinct roots.

This function is a polynomial and thus continuous everywhere. Plugging in values gives us $f(-2) = -1, f(-1) = 1, f(0) = 1, f(1) = -1, f(2) = 3$. Thus by the intermediate value theorem there is a root between -2 and -1, a root between 0 and 1, and a root between 1 and 2.

Example 4.26. Prove that every odd-degree polynomial has a root.

This is a bit trickier, but extremely useful! We know that all polynomials are continuous. We will see in the next section that every odd-degree polynomial will eventually take on very large values in one direction, and very negative values in the other direction. In particular, every odd-degree polynomial outputs a positive output for some input, and a negative output for some input. Thus by the Intermediate Value Theorem, it must have 0 as an output for some input.

4.3 The exponential and the logarithm

In this section we'll look at a specific, extremely important example: the exponential function $e^x = \exp(x)$ and its inverse the logarithm.

In the first weeks of the course, we discussed the power function $f(x) = x^n$. It's simple to define x^n when n is a positive integer, as $x \cdot x \cdot \dots \cdot x$. It's now clear that we defined $x^{1/n}$ as the inverse function to x^n, with domain restricted to positive numbers in the case n is even and thus x^n is not one-to-one. But can we make sense of x^r where r is any real number? What would it mean to write $2^{\sqrt{2}}$?

The answer would presumably be between 2 and 4. And also between $2^{1.4}$ and $2^{1.5}$. And between $2^{1.41}$ and $2^{1.42}$. In fact, this is how we will define $2^{\sqrt{2}}$. It turns out that there will be exactly one number greater than $2^1, 2^{1.4}, 2^{1.41}, 2^{1.414}, 2^{1.4142}, \dots$ and less than $2^2, 2^{1.5}, 2^{1.42}, 2^{1.415}, 2^{1.4143}, \dots$.

And if this sounds like the approximation-by-zooming in we did with the intermediate value theorem, you're right! If x is a rational or decimal approximation to the real number r, then 2^x should be an approximation to 2^r, and as x gets closer to r the approximation should get better. Thus we get the following definition:

Definition 4.27. If r is any real number, and a is a positive real number, we define $a^r = \lim_{x \to r} a^x$ for x varying over the rational numbers. We say that a is the *base* and r is the *exponent*.

Remark 4.28. We can't actually raise a negative real number to an irrational power. The limit would vary over x with even denominator, and a^x is not defined if x has even denominator and $a < 0$.

Proposition 4.29. *The exponential function $f_a(x) = a^x$ is well-defined for any r when $a > 0$, and is continuous on all real numbers. Further, it satisfies the exponential laws:*

- $a^{x+y} = a^x a^y$
- $a^{x-y} = \frac{a^x}{a^y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$.

Proposition 4.30. *If $a > 1$, then $\lim_{x\to+\infty} a^x = +\infty$ and $\lim_{x\to-\infty} a^x = 0$.*

If $0 < a < 1$ then $\lim_{x\to+\infty} a^x = 0$ and $\lim_{x\to-\infty} a^x = +\infty$.

Proof. Both of these can be seen by considering cases where x is an integer. □

In section 3.4 we saw that the number

$$e = \lim_{x\to 0}(1+x)^{1/x} \approx 2.71828$$

naturally appears in certain differential equations that lead to exponential growth. We saw that $\frac{d}{dx}e^x = e^x$, and we were able to compute some derivatives from that. But what if we want to compute $\frac{d}{dx}2^x$? There are two different ways to approach this calculation, but they both require the logarithm.

4.3.1 Logarithms

The exponential function $f(x) = a^x$ is one-to-one, since if $f(x) = f(y)$, then $a^x = a^y$, which means that $a^{x-y} = 1$ and so $x - y = 0$. So a^x must have an inverse function.

Definition 4.31. The *logarithmic function with base a*, written $\log_a$, is the inverse function to a^x. It has domain $(0, +\infty)$, and its image is all real numbers. We often write ln for $\log_e$.

Thus if $a > 0$, we see that $\log_a(a^x) = x$ for every real x, and $a^{\log_a(x)} = x$ for every $x > 0$.

Example 4.32.

- $\log_3(9) = 2$.
- $\log_2(8) = 3$
- $\log_a(1) = 0$ for any $a > 0$.

Proposition 4.33. *If $a > 1$, then* $\lim_{x\to+\infty} \log_a(x) = +\infty$ *and* $\lim_{x\to 0^+} \log_a(x) = -\infty$.

The logarithm also has a number of properties corresponding to the exponential laws:

Proposition 4.34.

- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$
- $\log_a(x^r) = r \log_a(x)$ *for any real number* r.

Example 4.35.

- $\ln(a) + \frac{1}{2}\ln(b) = \ln(a) + \ln(b)^{1/2} = \ln(a\sqrt{b})$.
- Solve $e^{5-3s} = 10$. We have that $5 - 3x = \ln 10$ and so $x = \frac{5-\ln 10}{3}$.

Remark 4.36. These properties are actually historically why the logarithm was originally important. Before calculators, people doing difficult computational work had to work by hand. Adding five digit numbers is much, much easier than multiplying them. So engineers would take the log of the numbers, add them together, and then exponentiate. This was all done with the help of massive books called log tables that would tell you the logarithm of a given number. Slide rules are essentially a way of making the log tables portable; but they were superseded by pocket calculators.

There is one more important logarithmic formula, corresponding to the exponential law I left out:

Proposition 4.37 (change of base)**.** *For any positive number $a \neq 1$, we have* $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$.

Proof. $\exp(\log_a(x) \cdot \ln(a)) = a^{\log_a(x)} = x$, so $\log_a(x) \cdot \ln(a) = \ln(x)$. ☐

This allows us to convert logs in any base to logs in another base.

Example 4.38. What is $\log_2 10$? By the change of base formula, we have $\log_2(10) = \frac{\ln 10}{\ln 2}$. $\ln 10 \approx 2.3$ and $\ln 2 \approx .7$, so $\log_2 10 \approx 2.3/.7 \approx 23/7$.

4.3.2 Derivatives of exponentials and logs

Now we're ready to start computing derivatives. Recall that $e = \lim_{x\to 0}(1 + x)^{1/x}$.

Proposition 4.39. *The function* $f(x) = \log_a(x)$ *is differentiable, with derivative* $f'(x) = \frac{1}{x}\log_a e$.

Proof.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\log_a(x+h)-\log_a(x)}{h} \\
&= \lim_{h\to 0} \frac{\log_a((x+h)/x))}{h} \\
&= \lim_{h\to 0} \frac{\log_a(1+\frac{h}{x})}{h} \\
&= \frac{1}{x}\lim_{h\to 0} \frac{x}{h}\log_a(1+\frac{h}{x}) \\
&= \frac{1}{x}\lim_{h\to 0} \log_a\left((1+\frac{h}{x})^{x/h}\right) \\
&= \frac{1}{x}\log_a\left(\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{1}{h/x}}\right) \\
&= \frac{1}{x}\log_a(e)
\end{aligned}$$

□

Corollary 4.40. *If* $f(x) = \log_a(x)$ *then* $f'(x) = \frac{1}{x \ln a}$.

Proof. By the change of base formula, $\log_a(e) = \frac{\ln(e)}{\ln(a)}$. □

Corollary 4.41. $\ln'(x) = \frac{1}{x}$.

Remark 4.42. An alternate path to discover the natural logarithm is to ask "what is the function whose derivative is $1/x$?" We will return to this line of thought in Lab 8.

Example 4.43.

- Let $f(x) = \ln(x^3+1)$. Then $f'(x) = \frac{1}{x^3+1} \cdot 3x^2$.
- Let $g(x) = \log_a(\cos(x))$. Then $g'(x) = \frac{1}{\cos(x)\ln(a)} \cdot (-\sin(x)) = -\tan(x)/\ln(a)$.
- If $h(x) = \ln(|x|)$ then $h'(x) = 1/x$ if $x > 0$ and $h'(x) = (-1/x)\cdot(-1) = 1/x$ if $x < 0$. So $h'(x) = \frac{1}{x}$.

We can sometimes use logarithms and implicit differentiation to make difficult differentiation problems easier, just as we use them to simplify difficult arithmetic problems.

Example 4.44 (Power Rule)**.** If r is a real number and $f(x) = x^r$, then

$$\begin{aligned} y &= x^r \\ \ln|y| &= r\ln|x| \\ \frac{1}{y}\frac{dy}{dx} &= r\frac{1}{x} \\ \frac{dy}{dx} &= r\frac{y}{x} = rx^{r-1}. \end{aligned}$$

And finally, we can use the logarithmic derivatives to figure out the derivative of $\exp_a$.

Proposition 4.45. *If $f(x) = a^x$ for $a > 0$, then f is differentiable and $f'(x) = a^x \ln a$.*

Proof.

$$\begin{aligned} y &= a^x \\ \ln|y| &= x\ln|a| \\ \frac{1}{y}\frac{dy}{dx} &= \ln a \\ \frac{dy}{dx} &= y\ln a = a^x \ln a. \end{aligned}$$

□

Corollary 4.46. $\exp'(x) = \exp(x)$.

Example 4.47.

- If $f(x) = e^{\sin(x)}$ then $f'(x) = e^{\sin(x)} \cdot \cos(x)$.
- If $g(x) = 5^{x^2+1}$ then $g'(x) = \ln(5)5^{x^2+1} \cdot 2x$.

Remark 4.48. An alternate approach here is to notice that $a^x = e^{x\ln a}$ so $\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln(a)} = e^{x\ln(a)} \cdot \ln(a) = a^x \ln(a)$. If this thought process is more comfortable for you, go ahead and use it instead.

Poll Question 4.3.1. If $h(x) = x^x$ we have to be *very careful*—the obvious approaches don't actually work. But logarithmically:

$$\begin{aligned} y &= x^x \\ \ln|y| &= x\ln|x| \\ \frac{1}{y}\frac{dy}{dx} &= \ln|x| + \frac{x}{x} = \ln|x| + 1 \\ \frac{dy}{dx} &= x^x(\ln|x| + 1). \end{aligned}$$

So $h'(x) = (\ln|x| + 1)x^x$.

You can get the same result by writing $h(x) = e^{x\ln(x)}$, and thus $h'(x) = e^{x\ln(x)}(\ln(x)+1) = x^x(\ln(x)+1)$.

Example 4.49. We wish to find the derivative of $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

$$\begin{aligned}
\ln y &= \frac{3}{4}\ln(x) + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{3}{4x} + \frac{2x}{2x^2+2} - \frac{3\cdot 5}{3x+2} \\
\frac{dy}{dx} &= y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right) \\
&= \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right).
\end{aligned}$$

4.4 Inverse Trigonometric Functions

We can invert some polynomials, and we can invert exponential functions. The other very common sort of function to work with is a trigonometric function, and we'd like to find inverses to these as well.

As a straightforward question, we cannot invert the trigonometric functions because they are all periodic, and thus not one-to-one. For instance, $\sin(0) = \sin(\pi) = \sin(2\pi) = \sin(n\pi)$ for any integer n.

However, sometimes a function is invertible if you restrict its domain enough, e.g. to between two critical points. In this section we make canonical domain choices for the trigonometric functions such that they are invertible.

Definition 4.50. If $-1 \leq x \leq 1$, we define $\arcsin(x) = \sin^{-1}(x) = y$ where $\sin(y) = x$ and $-\pi/2 \leq y \leq \pi/2$.

arcsin has a domain of $[-1,1]$ and a range of $[-\pi/2, \pi/2]$.

Example 4.51. We can determine that $\arcsin(-\sqrt{3}/2) = -\pi/3$ since $\sin(-\pi/3) = -\sqrt{3}/2$. (Of course, $\sin(5\pi/3) = -\sqrt{3}/2$ as well, but $5\pi/3 > \pi/2$).

With more cleverness, we can calculate $\cos(\arcsin(1/3))$. Suppose $\theta = \arcsin(1/3)$. Then θ is the angle of a triangle with opposite side of lenght 1 and hypotenuse of length 3; using the Pythagorean theorem we determine that the other side has length $\sqrt{8} = 2\sqrt{2}$. Since $\cos(\theta)$ is the length of the adjacent side over the hypotenuse, we have $\cos(\arcsin(1/3)) = 2\sqrt{2}/3$.

We can make similar definitions for inverse cosine and inverse tangent functions. We do have to be careful about the precise domains and images.

Definition 4.52. If $-1 \leq x \leq 1$, we define $\arccos(x) = \cos^{-1}(x) = y$ where $\cos(y) = x$ and $0 \leq y \leq \pi$. This function has domain $[-1, 1]$ and range $[0, \pi]$.

If x is a real number, we define $\arctan(x) = \tan^{-1}(x) = y$ where $\tan(y) = x$ and $-\pi/2 < y < \pi/2$. This function has domain $(-\infty, +\infty)$ and image $(-\pi/2, \pi/2)$.

$\lim_{x\to+\infty} \arctan(x) = \pi/2$ and $\lim_{x\to-\infty} \arctan(x) = -\pi/2$.

sin and cos and tan are all differentiable functions, so by the Inverse Function Theorem, so are arcsin and arccos and arctan, at least most of the time.

Proposition 4.53.

- $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$
- $\arccos'(x) = \frac{-1}{\sqrt{1-x^2}}$
- $\arctan'(x) = \frac{1}{1+x^2}$.

Proof. There are two approaches to proving these facts. One involves trigonometric identities, and the other involves thinking about triangles. They both involve implicit differentiation.

Suppose $y = \arcsin(x)$. Then $\sin(y) = x$ and thus $\cos(y)\frac{dy}{dx} = 1$. Then we have $\frac{dy}{dx} = \frac{1}{\cos(y)}$.

From here, we can say two things. One is that $\cos(y) = \sqrt{1 - \sin^2(y)} = \sqrt{1 - x^2}$, using the trigonometric identity that $\cos^2(y) + \sin^2(y) = 1$ and being careful about sign choices.

I find it easier to think the following thing: if $y = \arcsin(x)$ then y is the angle of a triangle where the opposite side has length x and the hypotenuse has length 1. Then the other side has length $\sqrt{1 - x^2}$, so $\cos(y) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1 - x^2}$.

Note we got the same answer both ways, and they both involved basically the same facts; the identity $\sin^2(y) + \cos^2(y) = 1$ holds precisely because of the triangle argument. Either way you want to think of it is fine with me.

We can do the same with $\arccos(x)$. $\cos(y) = x$, so $\frac{dy}{dx} = \frac{-1}{\sin(y)} = -\frac{1}{\sqrt{1-x^2}}$.

arctan is slightly trickier. $\tan(y) = x$ so $\sec^2(y)\frac{dy}{dx} = 1$, and thus we have $\frac{dy}{dx} = \cos^2(y)$. Again, we can use the identity $1 + \tan^2(y) = \sec^2(y)$, but if we don't remember that we can see that y is the angle of a triangle with opposite side x and adjacent side 1, and hence hypotenuse $\sqrt{1 + x^2}$. Then $\cos(y) = \frac{1}{\sqrt{1+x^2}}$ and so $\arctan'(x) = \cos^2(y) = \frac{1}{1+x^2}$. □

Example 4.54. What is $\arcsin'(.75)$? $\frac{1}{\sqrt{1-9/16}} = \frac{1}{\sqrt{7/16}}$.

What is $\arctan'(e^x)$? $\frac{1}{1+e^{2x}} \cdot e^x$.

What is $\arccos'(x^2 + 2x + 3)$? $\frac{1}{\sqrt{1-(x^2+2x+3)^2}} \cdot (2x + 2)$.

4.5 L'Hôpital's Rule

We often find ourselves wanting to evaluate limits of "indeterminate form": that is, the limit of a quotient whose numerator and denominator both approach 0 or both approach $\pm\infty$. In the past we've used various tricks to work out such limits, but today we develop a new and widely-applicable tool. This tool is especially useful for dealing with limits involving ln or exp.

Theorem 4.55 (L'Hôpital's Rule)**.** *Suppose f and g are differentiable, and $g'(x) \neq 0$ near a, except possibly at a. Suppose either $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$. (In other words, the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate form). Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right exists.

Remark 4.56. Note that L'Hôpital's Rule *only* applies to limits of indeterminate form.

Proof. We won't prove this fully, but we will prove it in the case where $f(a) = g(a) = 0$, $g'(a) \neq 0$, and f' and g' are continuous at a.

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} \\
&= \lim_{x\to a} \frac{(f(x) - f(a))(x-a)}{(g(x) - g(a))(x-a)} \\
&= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} \\
&= \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.
\end{aligned}$$

□

Example 4.57.

$$\begin{aligned}
\lim_{x\to 3} \frac{x^2 - 4x + 3}{x^2 - 2x - 3} &= \lim_{x\to 3} \frac{2x-4}{2x-2} = \frac{2}{4} = \frac{1}{2}. \\
\lim_{x\to 0} \frac{1 - \cos(x)}{\sin(x)} &= \lim_{x\to 0} \frac{\sin(x)}{\cos(x)} = \frac{0}{1} = 0. \\
\lim_{x\to 1} \frac{\ln x}{x-1} &= \lim_{x\to 0} \frac{1/x}{1} = 1.
\end{aligned}$$

Sometimes we have to apply L'Hôpital's rule more than once to get the results we want.

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x - x}{x^3} &= \lim_{x\to 0}\frac{\sec^2(x)-1}{3x^2}\\ &= \lim_{x\to 0}\frac{2\sec^2(x)\tan(x)}{6x} = \lim_{x\to 0}\frac{\tan x}{3x}\\ &= \lim_{x\to 0}\frac{\sec^2(x)}{3} = \frac{1}{3}.\end{aligned}$$

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2} = \lim_{x\to 0}\frac{e^x-1}{2x} = \lim_{x\to 0}\frac{e^x}{2} = \frac{1}{2}.$$

We can also use L'Hôpital's rule to evaluate limits at infinity.

Example 4.58.

$$\begin{aligned}\lim_{x\to\pm\infty}\frac{x^2+5x+3}{x^2+7x-2} &= \lim_{x\to\pm\infty}\frac{2x+5}{2x+7}\\ &= \lim_{x\to\pm\infty}\frac{2}{2} = 1.\\ \lim_{x\to+\infty}\frac{\ln(x)}{x} &= \lim_{x\to+\infty}\frac{1/x}{1} = 0.\\ \lim_{x\to+\infty}\frac{e^x}{x} &= \lim_{x\to+\infty}\frac{e^x}{1} = +\infty.\end{aligned}$$

In fact, it's not too hard to see, using L'Hôpital's Rule, that $\lim_{x\to+\infty}\frac{e^x}{x^n} = +\infty$ and $\lim_{x\to+\infty}\frac{\ln(x)}{x^n} = 0$.

Remember that L'Hôpital's rule only applies if we start with an indeterminate form.

Example 4.59.

$$\begin{aligned}\lim_{x\to\pi}\frac{\sin(x)}{1-\cos(x)} &\neq \frac{\cos(x)}{\sin(x)} = \pm\infty\\ \lim_{x\to\pi}\frac{\sin(x)}{1-\cos(x)} &= \frac{0}{1-(-1)} = 0.\end{aligned}$$

A more dangerous example:

$$\lim_{x\to 0}\frac{e^x-1-x}{x^3} = \lim_{x\to 0}\frac{e^x-1}{3x^2} = \lim_{x\to 0}\frac{e^x}{6x}$$

You might think we should use L'Hôpital's rule again here; that would give $\lim_{x\to 0}\frac{e^x}{6} = 1/6$. But the top goes to 1 and the bottom goes to 0, so this is not an indeterminate form! The true limit is $\pm\infty$.

And sometimes L'Hôpital's rule doesn't alwasy work the way we'd like it to, just "because it doesn't."

Example 4.60.

$$\lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\pm\infty} \frac{1}{\frac{x}{\sqrt{x^2+1}}} = \lim_{x\to\pm\infty} \frac{\sqrt{x^2+1}}{x}$$

But here if we're clever we can observe that if the limit exists, then

$$\left(\lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}}\right)^2 = 1$$
$$\lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}} = \pm 1.$$

We can often use L'Hôpital's rule to compute limits of other indeterminate forms with a bit of cleverness. Recall the "minor" indeterminate forms are $1^\infty, \infty - \infty, 0^0, \infty^0, 0 \cdot \infty$. Products can obviously be rewritten as quotients, and sums or differences can often be combined into something by collecting common denominators. Exponents can be turned into ratios by means of logarithms.

Example 4.61.

$$\begin{aligned}
\lim_{x\to\pi/2} \sec(x) - \tan(x) &= \lim_{x\to\pi/2}\left(\frac{1}{\cos(x)} - \frac{\sin(x)}{\cos(x)}\right) \\
&= \lim_{x\to\pi/2} \frac{1-\sin(x)}{\cos(x)} \\
&= \lim_{x\to\pi/2} \frac{-\cos(x)}{-\sin(x)} = \frac{0}{1} = 0. \\
\lim_{x\to 0} \cot(2x)\sin(6x) &= \lim_{x\to 0} \frac{\sin(6x)\cos(2x)}{\sin(2x)} = 1 \cdot \lim_{x\to 0} \frac{\sin(6x)}{\sin(2x)} \\
&= \lim_{x\to 0} \frac{6\cos(6x)}{2\cos(2x)} = 3. \\
\lim_{x\to 1} x^{1/(1-x)} &= \exp\left(\lim_{x\to 1} \frac{\ln x}{1-x}\right) \\
&= \exp\left(\lim_{x\to 1} \frac{1/x}{-1}\right) = \lim_{x\to 1} e^{-1/x} = 1/e.
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to+\infty} x^{1/x} &= \exp\left(\lim_{x\to+\infty} \frac{\ln x}{x}\right) \\
&= \exp\left(\lim_{x\to+\infty} \frac{1/x}{1}\right) \\
&= \exp(0) = 1. \\
\lim_{x\to 0^+} x^{1/\ln(x)} &= \exp\left(\lim_{x\to 0^+} \frac{\ln(x)}{\ln(x)}\right) \\
&= \exp\left(\lim_{x\to 0^+} 1\right) = e.
\end{aligned}$$