Lab 5 Tuesday September 24

Linear Approximation

We know that if we have a function $f(x)$ and know what it looks like at a point a, we can use the derivative to give a linear approximation

$$f(x) \approx f(a) + f'(a)(x-a).$$

Last lab, we drew secant lines, which are lines that intersect the graph of a function in (at least) two points; we may recall that by rearranging some information, we can write

$$f(x) = \frac{f(x) - f(a)}{x-a}(x-a) + f(a).$$

As x approaches a, this becomes closer to being a tangent line, and the slope term becomes closer to $f'(a)$. Thus we can get a decent approximation, if x and a are close, by replacing this difference quotient with the derivative:

$$f(x) \approx f'(a)(x-a) + f(a).$$

In this lab we want to push that idea a bit farther and see what we can do with it—and when it breaks down.

Error Margins

Now answer the following questions. In each problem, before you do any computations, think carefully about what you should use for f, a, b.

1. Estimate $(2.1)^5$ without doing any calculations. Then use the tangent line to approximate $(2.1)^5$. Finally, use Mathematica to compute the exact answer. How far off were you?

   Solution: The obvious rough estimate is $2^5 = 32$.

   We take $f(x) = x^5$ and $a = 2$. Then $f'(x) = 5x^4$, so we have $f(2) = 32, f'(2) = 80$, and

   $$f(2.1) \approx 80(2.1-2) + 32 = 40.$$

   The exact answer is 40.841. We can see our original estimate as the constant term in our tangent line approximation.

2. Now approximate $(2.5)^5$ using $a = 2$. Approximate 3^5 using $a = 2$. Are your approximations getting better or worse? Why? What does this tell you about what counts as "close" to 2?

   Solution: We have

   $$\begin{aligned}(2.5)^5 &\approx 80 \cdot (2.5-2) + 32 = 72\\ 3^5 &\approx 80 \cdot (3-2) + 32 = 112.\end{aligned}$$

   The true answers are 97.6563 and 243. Unlike in part (a), these estimates are not especially good. This is because 3 is actually not very close to 2—especially proportionately. Of course, it's not that hard to compute 3^5 directly.

   These methods are best when $x - a$ is very small relative to everything else. We often use them in the real world for $x - a < .1$ or so.

3. Without calculating, find an upper bound and a lower bound for $(4.5)^3$. (Hint: $4 < 4.5 < 5$). Now approximate 4.5^3 with a tangent line in two different ways, from two different base points (that is, two different choices of a). What happens?

   Solution: For our bounds, we would expect $64 = 4^3 < 4.5^3 < 5^3 = 125$.

   We take $f(x) = x^3$ and $a = 4$. Then $f(a) = 64$, and $f'(x) = 3x^2$ so $f'(a) = 48$, and

$$f(4.5) \approx 48(4.5 - 4) + 64 = 24 + 64 = 88.$$

   Alternatively, we can take $f(x) = x^3$ and $a = 5$. Then $f(a) = 125, f'(x) = 3x^2, f'(a) = 75$, and

$$f(4.5) \approx 75(4.5 - 5) + 125 = 125 - 37.5 = 87.5.$$

   The exact answer is 91.125. These approximations are both decent but not great—as we'd expect, since 4.5 is close-ish to 4 and to 5, but not especially close.

4. Approximate `CubeRoot[28]` and `82^(1/4)`.

   Solution: We take $a = 27$ and $a = 81$ respectively.

$$\sqrt[3]{28} \approx \frac{1}{3}(27)^{-2/3}(28 - 27) + 3 = \frac{1}{27} + 3 \approx 3.03704$$
$$\sqrt[4]{82} \approx \frac{1}{4}(81)^{-3/4}(82 - 81) + 3 = \frac{1}{108} + 3 \approx 3.00926.$$

   The true answers are approximately 3.03659 and 3.00922 respectively.

5. Now approximate `28^3` and `82^4` using the same base points a that you used in the last problem. Are these approximations better or worse than your approximations of `CubeRoot[28]` and `82^(1/4)` above? Why? Would you do this on your own?

   Solution: We have

$$28^3 \approx 3(27)^2(28 - 27) + 27^3 = 21870$$
$$82^4 \approx 4(81)^3(82 - 81) + 81^4 = 45172485$$

   In contrast the true answers are 21952 and 45172485.

   These approximations aren't *terrible* but they aren't very good either. Since the derivative is changing quickly here (the second derivatives are $6 \cdot 27$ and $12 \cdot 81^2$ respectively), the approximation won't be very good.

6. If you take $a = 0$ and $f(x) = x^{10}$, use a tangent line to approximate $f(2)$. What happens and why? What if you instead approximate with $a = 1$?

   Solution: We have $f'(x) = 10x^9$, so we have $f'(0) = 0$, and thus

$$f(2) \approx 0(2 - 0) + 0 = 0.$$

   If we take $a = 1$, we have

$$f(2) \approx 10(2 - 1) + 1 = 11.$$

   The true answer is 1024, which is far away from both of those. In essence, the derivative is changing so quickly that the tangent line approximation is not very good over those distances.

Finding Formulas

Sometimes rather than just approximating a specific number, we want to actually use a linear formula to approximate our function. For each of these problems you should find a formula for the linear approximation. Then graph your approximation and your original function together and see how they compare.

Here are some derivative rules we'll discuss in class over the next week that will be helpful for this worksheet:

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\tan(x) = \sec^2(x) \qquad \frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x) \qquad \frac{d}{dx}\cot(x) = -\csc^2(x) \qquad \frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$
$$\frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2} \qquad \frac{d}{dx}(1+x)^\alpha = \alpha(1+x)^{\alpha-1}$$

1. Approximate `Sin[.05]` and `Cos[.05]` Note that this is .05 and not .5.

   Solution: We take $a = 0$. Then since $\sin'(x) = \cos(x)$ and so $\sin'(0) = \cos(0) = 1$, we have
   $$\sin(.05) \approx 1(.05 - 0) + 0 = .05.$$
   The true answer is about .04998.

   Similarly, $\cos'(x) = -\sin(x)$ so $\cos'(0) = 0$. Then
   $$\cos(.05) \approx 0(.05 - 0) + 1 = 1.$$
   The true answer is about .9986.

2. Find a formula to approximate $\sin(x)$ when x is "small". (This is the revenge of the Small Angle Approximation). Find a formula to approximate $\cos(x)$ when x is small. What's unusual about this second formula?

   Solution: We can use our work from the last problem. We get $\sin(x) \approx x$ for small x; this is basically the small angle approximation.

   For $\cos(x)$, we know that $\cos(0) = 1$ and $\cos'(0) = 0$, so we get $\cos(x) \approx 1$ for small x. This approximation is a little awkward because we have cosine being approximately constant, despite the fact that it obviously isn't actually constant.

3. Find formulas to approximate $\tan(x)$ and $\sec(x)$ near $a = 0$. How do these formulas relate to the ones you worked out for sin and cos?

   Solution: We can work these out directly. We get
   $$\tan(x) \approx \tan(0) + \sec^2(0)(x - 0) = x$$
   $$\sec(x) \approx \sec(0) + \sec(0)\tan(0)(x - 0) = 1.$$
   We could also see this another way. We know that $\sin(x) \approx x$ and $\cos(x) \approx 1$, so
   $$\tan(x) = \frac{\sin(x)}{\cos(x)} \approx \frac{x}{1} = x.$$
   Similarly, we know that $\sec(x) = \frac{1}{\cos^2(x)}$ so $\sec(x) \approx \frac{1}{1^2} = 1$.

4. Find formulas to approximate $\sin(x)$ and $\cos(x)$ near $a = \pi/2$. How do these relate to the formulas from number 2?

   Solution: We get

$$\sin(x) \approx \sin(\pi/2) + \cos(\pi/2)(x - \pi/2) = 1$$
$$\cos(x) \approx \cos(\pi/2) - \sin(\pi/2)(x - \pi/2) = -(x - \pi/2) = \pi/2 - x.$$

   These formulas are similar to the formulas near 0, but $\cos(x)$ and $\sin(x)$ have sort of swapped places. We also get a negative coefficient in the $\cos(x)$ approximation, from the formula for the derivative of $\cos(x)$.

5. Find formulas to approximate $\sin(x)$ and $\cos(x)$ near $a = \pi/4$. What do you notice here?

   Solution: Here we get

$$\sin(x) \approx \sin(\pi/4) + \cos(\pi/4)(x - \pi/4) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \pi/4)$$
$$\cos(x) \approx \cos(\pi/4) - \sin(\pi/4)(x - \pi/4) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x - \pi/4).$$

   Except for the sign, these formulas are exactly the same—precisely because $\cos(x)$ and $\sin(x)$ are doing basically the same thing at $\pi/4$.

6. Can you find formulas to linearly approximate $\cot(x)$ and $\csc(x)$ near $a = 0$?

   Solution: No, because neither function is defined at 0.

7. Find a formula to approximate $f(x) = x^3 + 3x^2 + 5x + 1$ near $a = 0$. What do you notice? Why does that happen?

   Solution: We have $f(0) = 1$ and $f'(x) = 3x^2 + 6x + 5$ so $f'(0) = 5$. Thus

$$f(x) \approx 1 + 5x.$$

   This is exactly what you get if you take the original polynomial and cut off all the terms of degree higher than 1.

   This makes sense, because we're looking for the closest we can get to f without using terms of degree higher than 1.

8. Find a formula to linearly approximate $f(x) = \dfrac{1}{1-x}$ near $x = 0$. (Hint: we can do this without the chain rule using the quotient rule).

   Solution: We compute that $f'(x) = \frac{0(1-x)-(-1)(1)}{(1-x)^2} = \frac{1}{(1-x)^2}$, so $f'(0) = 1$. Then

$$f(x) \approx 1 + x.$$

   This is a special case of what's known as the geometric series formula.

9. Can we linearly approximate $f(x) = 1/x$ near 0?

   Solution: No. We see that f is undefined at 0. More importantly, $f'(x) = -1/x^2$ is also undefined at zero. So there's no linear approximation.

10. Can we linearly approximate $f(x) = 1/x$ near 1?

    Solution: Yes! We have $f(1) = 1$ and $f'(1) = -(1)^{-2} = -1$ so

    $$f(x) \approx 1 - (x-1) = 2 - x.$$

11. Approximate $(1.01)^{10}$.

    Solution: Our function is $f(x) = x^{10}$ and our $a = 1$. So $f(a) = 1$ and $f'(a) = 10a^9 = 10$. Then we have

    $$f(1.01) \approx 10(1.01 - 1) + 1 = 1.1.$$

    The true answer is about 1.10462.

12. Approximate $(1.01)^\alpha$ where $\alpha \neq 0$ is some constant (your answer will have an α in it).

    Solution: We have $f(x) = x^\alpha$, so $f'(x) = \alpha x^{\alpha-1}$. We again have $f(1) = 1$ and $f'(1) = \alpha(1)^{\alpha-1} = \alpha$, so

    $$f(1.01) \approx \alpha(1.01 - 1) + 1 = 1 + \alpha/100.$$

13. Now find a formula to approximate $(1 + x)^\alpha$ where x is "small" and $\alpha \neq 0$ is a constant. (This rule is the called the "binomial approximation" and is often useful in physics).

    Solution: We still take $f(x) = x^\alpha$ and $a = 1$. But we compute

    $$f(1 + x) \approx 1 + \alpha(1 + x - 1) = 1 + \alpha x.$$

    This formula is used constantly in physics and other applications.

    It is probably more helpful in the long run to think about $f(x) = (1 + x)^\alpha$, though. Then we have $f'(x)_= \alpha(1 + x)^{\alpha-1}$ (which result requires the chain rule), and then we get

    $$f(x) \approx 1 + \alpha x.$$

14. Bonus: Can you find a formula to approximate $(1 + x^n)^\alpha$ when x is small?

    Solution: There are two different things we could do here.

    One uses the chain rule, which we will actually cover in class tomorrow. If $f(x) = (1+x^n)^\alpha$, it's not too hard to work out that $f'(x) = \alpha(1 + x^n)^{\alpha-1} \cdot nx^{n-1}$ and thus $f'(0) = 0$, so our tangent line approximation is

    $$f(x) \approx f(0) + f'(0)(x - 0) = 1 + 0(x - 0) = 0.$$

    So we see that $f(x) \approx 1$. This isn't very helpful!

    However, we could do something completely different. If we take $g(x) = (1 + x)^\alpha$, then we have $f(x) = g(x^n)$. But if x is close to 0, then x^n will be even closer to zero. So we have

    $$f(x) = g(x^n) \approx 1 + \alpha x^n.$$

    But that raises another question. Why do we get two genuinely different answers here?