# Lab 6 Tuesday, October 8

## Euler's Method

### Comparing with true solutions

Download the file `euler.nb` from the course web page, and evaluate the first block of code. This will give you four functions.

`euler[df,x0,y0,xfin,n]` uses Euler's method to approximate $f(x_{fin})$ given the initial data $y' = df, f(x_0) = y_0$ and $n$ steps. It outputs the estimate of $f(x_{fin})$.

`eulerplot[df,x0,y0,xfin,n]` runs the same approximation, but instead of reporting the estimate as a number, it plots the data points.

`comparewitherrors[f_, df_, x0_, y0_, xfin_, n_]` takes a known function and plots that function on the domain $[x_0, x_{fin}]$; against it it plots the data points generated by using Euler's method to estimate $f(x_{fin})$ with the initial data $f(x_0) = y_0$.

1. Consider the function $f(x) = e^x$, which satisfies $y' = y$.
   (a) Use the function `comparewitherrors[E^x,y, 0,1,5,5]` to compare the actual function to the results of Euler's method.
   (b) Why did we pick the initial values we did?
   (c) Try again with 10 steps instead of 5. Try 100 steps. Play around and see what happens as you change the step size.

2. Another important differential equation is the *logistic growth equation*, which is often used to model population growth. The differential equation is $y' = y(1 - y)$ and the corresponding function is $f(x) = \frac{1}{1+\frac{1-x_0}{x_0}e^{-x}}$ or `L[x_]:=1 / (1 + (1-x0)/x_0 E^(-x))`, where $x_0$ is the initial condition
   (a) We'll take the initial condition $x_0 = 1/2$. (This corresponds to a population at 50% of maximum capacity). This gives the function `L[x_]:=1/(1 + E^(-x))`
   (b) Use the command `compareplot[1/(1+E^(-x)),0,.5,10,10]` to estimate $L(10)$. What happens?
   (c) What if we raise the step size to 100?
   (d) Now let's compare to the differential equation. Use the command `comparewitherrors[1/(1 + E^(-x)), y (1 - y), 0, .5, 10, 10]` to use Euler's method to fit the equation. What happens? How is this different from before?
   (e) Increase the step size to 100.

3. Let's look at the same function with different initial conditions.
   (a) We'll take the initial condition $x_0 = 2$, corresponding to a population at double capacity. This gives the function `L[x_] := 1/(1 - E^(-x)/2)`.
   (b) Use the command `compareplot[1/(1-E^(-x)/2),0,2,10,10]` to estimate $L(10)$. What happens?
   (c) What if we raise the step size to 100? To 1000?

(d) Now let's compare to the differential equation. Use the command `comparewitherrors[1/(1 - E^(-x)/2), y (1 - y), 0, 2, 10, 10]` to use Euler's method to fit the equation. What happens? How is this different from before?

(e) Increase the step size to 100. To 1000.

(f) Does using the "true" derivative, or the differential equation, work better? Why do you think this is?

4. The differential equation $y' = y - e^x \sin(5x)/2 + 5e^x \cos(5x)$ with initial conditions $y(0) = 0$ has solution $y = e^x \sin(5)$.

   (a) Run `comparewitherrors[E^x Sin[5x],y-E^x Sin[5x]/2+5E^x Cos[5x],0,0,10,10]`

   (b) Try with 100 steps. Try with 1000.

5. If we take $y' = 2y/x - x^2y^2$, you can check that $f(x) = \frac{5x^2}{x^5+4}$ is a solution, with initial condition $f(1) = 1$.

   (a) Use the command `comparewitherrors[5 x^2/(x^5 + 4), 2 y/x - x^2 y^2, 1, 1, 5, 10]` to use Euler's method to fit the equation. What happens? How is this different from before?

   (b) Increase the number of steps to 100. To 1000.

**Euler's Method Estimations**

To learn to use the `euler` command, we'll look at the example equation $f'(t) = f(t) - f(t)^2/2$ with $f(0) = 1$.

1. Use the command `euler[y - y^2/2, 0, 1, 3, 3]` to estimate $f(3)$ given $y' = y - y^2/2$ and $f(0) = 1$, with three steps.

2. Use the command `eulerplot[y - y^2/2, 0, 1, 3, 3]` to see these results graphically.

3. Now try using nine steps, with `euler[y - y^2/2,0,1,3,9]`. What changes? Do the same with `eulerplot` . Now try using 100 steps.

For each of the remaining problems, do them by hand and use the computer to check your answers. See what happens if you increase the number of steps.

1. Suppose $f'(x) = xf(x)$, and $f(0) = 3$. Use four steps to estimate $f(4)$.

   **Solution:**

$$\begin{aligned} f(1) &\approx f'(0)(1-0) + f(0) = 0 \cdot 1 + 3 = 3 \\ f(2) &\approx f'(1)(2-1) + f(1) \approx (1 \cdot 3) \cdot 1 + 3 = 6 \\ f(3) &\approx f'(2)(3-2) + f(2) \approx (2 \cdot 6) \cdot 1 + 6 = 18 \\ f(4) &\approx f'(3)(4-3) + f(3) \approx (3 \cdot 18) \cdot 1 + 18 = 72. \end{aligned}$$

2. Suppose $f'(x) = e^x$ and $f(0) = 1$. Use four steps to estimate $f(4)$.

**Solution:**

$$\begin{aligned}
f(1) &\approx f'(0)(1-0)+f(0) = 1\cdot 1+1 = 2\\
f(2) &\approx f'(1)(2-1)+f(1) \approx e^1+2\\
f(3) &\approx f'(2)(3-2)+f(2) \approx e^2+e+2\\
f(4) &\approx f'(3)(4-3)+f(3) \approx e^3+e^2+e+2.
\end{aligned}$$

3. Suppose $f'(x) = f(x)$ and $f(0) = 1$. Use four steps to estimate $f(4)$.

   **Solution:**

$$\begin{aligned}
f(1) &\approx f'(0)(1-0)+f(0) = 1\cdot 1+1 = 2\\
f(2) &\approx f'(1)(2-1)+f(1) \approx 2\cdot 1+2 = 4\\
f(3) &\approx f'(2)(3-2)+f(2) \approx 4\cdot 1+4 = 8\\
f(4) &\approx f'(3)(4-3)+f(3) \approx 8\cdot 1+8 = 16
\end{aligned}$$

4. Suppose $f'(x) = \sin(f(x)) - x$ and $f(0) = 2$. Use three steps to estimate $f(\pi)$.

   **Solution:**

$$\begin{aligned}
f(\pi/3) &\approx f'(0)(\pi/3-0)+f(0) = (\sin(2)-0)(\pi/3-0)+2 \approx 2.95\\
f(2\pi/3) &\approx f'(\pi/3)(2\pi/3-\pi/3)+f(\pi/3) \approx (2.95-\pi/3)(\pi/3)+2.95 \approx 4.94\\
f(\pi) &\approx f'(2\pi/3)(\pi-2\pi/3)+f(2\pi/3) \approx (4.94-2\pi/3)(\pi/3)+4.94 \approx 7.92.
\end{aligned}$$

5. Suppose $f'(x) = f(x) - x$ and $f(1) = 3$. Use one step to estimate $f(2)$.

   Now use two steps to estimate $f(2)$. Now use four steps to estimate $f(2)$. What happens?

   **Solution:** One step:

$$f(2) \approx f'(1)(2-1)+f(1) = (3-1)(1)+3 = 5$$

   Two steps:

$$\begin{aligned}
f(3/2) &\approx f'(1)(3/2-1)+f(1) = (3-1)(1/2)+3 = 4\\
f(2) &\approx f'(3/2)(2-3/2)+f(3/2) \approx (4-3/2)(1/2)+4 = 21/4 = 5.25
\end{aligned}$$

   Four steps:

$$\begin{aligned}
f(5/4) &\approx f'(1)(5/4-1)+f(1) = (3-1)(1/4)+3 = 7/2\\
f(3/2) &\approx f'(5/4)(3/2-5/4)+f(5/4) \approx (7/2-5/4)(1/4)+7/2 = 65/16 \approx\\
f(7/4) &\approx f'(3/2)(7/4-3/2)+f(3/2) \approx \left(\frac{65}{16}-\frac{3}{2}\right)(1/4)+\frac{65}{16} = \frac{301}{64} \approx 4.0625\\
f(2) &\approx f'(7/4)(2-7/4)+f(7/4) \approx \left(\frac{301}{64}-\frac{7}{4}\right)(1/4)+\frac{301}{64} = \frac{1393}{256} \approx 5.44141.
\end{aligned}$$

   Each successive answer takes more work, but the answers are drifting upwards, suggesting all of them are underestimates converging to some higher answer.