Lab 8 Tuesday November 5

Antiderivatives

The last sort of inverse problem we'll discuss in this section, and in this course, is the inverse of differentiation.

Definition 0.1. If $F'(x) = f(x)$, we say that F is an *antiderivative* of f.

Notice that the definition says "an antiderivative." This is because our inverse isn't quite well-defined: If F is an antiderivative of f, then $F + 1$ is also an antiderivative, and in fact so is $F + C$ for any constant C. But it turns out that this is the only ambiguity possible.

Proposition 0.2. *Suppose F and G are both antiderivatives of f. Then $F(x) = G(x) + C$ for some constant C.*

Proof. $(F - G)'(x) = F'(x) - G'(x) = f(x) - f(x) = 0$. So $(F - G)'(x) = 0$ everywhere, and thus (by the Mean Value Theorem!) $(F - G)(x) = C$ is constant for some constant C. Thus $F(x) = G(x) + C$. □

Definition 0.3. If F is an antiderivative of f, we say that $F(x) + C$ is the *general form* of the antiderivative of f.

For each function $f(x)$ below, find a function $F(x)$ such that $F'(x) = f(x)$.

1. $3x^2$
2. x^2
3. $9e^{9x}$
4. e^{9x}
5. $\cos(x)$
6. $\cos(5x)$
7. $\sin(5x)$
8. $\dfrac{2}{1 + x^2}$
9. $\dfrac{2x}{1 + x^4}$
10. $\dfrac{2x}{1 + x^2}$
11. Near the surface of the earth, gravity accelerates objects at $-9.8m/s^2$. If an object is dropped from 100 meters up, what equation describes its velocity as a function of time? Its position? When does it hit the ground?
12. A car brakes with deceleration of $16ft/s^2$ and takes $200ft$ to come to a complete stop. How fast was the car travelling when it hit the brakes?

Numerical Integration

If we know $f(x)$, we can often figure out a function $F(x)$ by, essentially, guess and check. But we can also get an approximate value using a variant of Euler's method.

Suppose we know $f(x) = 1/x$ and we assume $F(1) = 0$, and we want to figure out what $F(2)$ is. We can use linear approximation:

$$F(2) \approx F(1) + f(1)(2-1) = 0 + 1 = 1.$$

If we want a better estimate, we can do linear approximation with more steps.

$$\begin{aligned} F(4/3) &\approx F(1) + f(1)(1/3) = 0 + 1/3 = 1/3 \\ F(5/3) &\approx F(4/3) + f(4/3)(1/3) \approx 1/3 + 1/4 = 7/12 \\ f(2) &\approx F(5/3) + f(5/3)(1/3) \approx 7/12 + 1/5 = 47/60. \end{aligned}$$

We can of course make our calculations more and more precise by taking more steps.

We can use code from the `euler.nb` file to see this more clearly. The command `eulerIntegral[1/x, 1, 0, 2, 3]` will use Euler's method on $f(x) = 1/x$, with $F(1) = 0$, to compute $F(2)$ using 3 steps. We can experiment to see what happens as the number of steps gets bigger.

If we know the antiderivative $F(x)$, we can also plot it for comparison. The command `eulerIntegralPlot[Log[x], 1, 0, 2, 3]` will plot this process for $F(x) = \ln(x)$ (which, we see, is an antiderivative of $1/x$ with $F(1) = 0$.) Again we can experiment with changing the step size to see what happens.

In each of the following problems, do the given number of steps of Euler's method by hand. Then use Mathematica to check your work and see what happens as the number of steps grows. Then find a formula for $F(x)$ and use `eulerIntegralPlot` to compare your estimates to the true $F(x)$.

1. If $F(1) = 1$ and $F'(x) = x^2$, use three steps to estimate $F(4)$.
2. If $F(0) = 1$ and $F'(x) = 2x$, use three steps to estimate $F(3)$.
3. If $F(0) = 0$ and $F'(x) = \frac{4}{1+x^2}$, use four steps to estimate $F(1)$. Do you notice something interesting about this value?

For the remaining problems, you won't be able to find an antiderivative; we've proven that there's no reasonable formula. So numeric techniques like this one are the only options we have. You should use a calculator or Mathematica to compute these integrals.

4. If $F(0) = 0$ and $F'(x) = \sqrt{1 - x^4}$, use three steps to estimate $F(1)$. This is called an elliptic integral, and is useful for computing the circumference of an ellipse.
5. If $F(0) = 1$ and $F'(x) = e^{-x^2}$, use three steps to estimate $F(1)$. This function is very important in the theory of statistics.
6. If $F(1) = 1$ and $F'(x) = \frac{1}{\ln(x)}$, use three steps to estimate $F(2)$.