Lab 9 Tuesday November 19

Quadratic Approximation

Exercises

For each exercise, plot the graphs of the true function, the linear approximation, and the quadratic approximation. Makes sure they match!

1. Use a quadratic approximation to estimate $\sqrt[3]{28}$.
2. If $f(x) = (x+8)^{1/2}$, compute a quadratic approximation centered at $a = 1$. Use this to estimate $f(1.02) = \sqrt{9.02}$.
3. If $f(x) = x^2 - 2x - 3$, compute a quadratic approximation centered at $a = 3$. How does this approximation compare to your original function?
4. Let $g(x) = x^4 - 3x^3 + 4x^2 + 4x - 2$. Compute the quadratic approximations at $a = 0$ and at $a = -2$. Compare them to $g(x)$. Estimate $g(-1.97)$.
5. Compute the quadratic approximations of $\sin(x)$ and $\cos(x)$ centered at zero. Estimate $\sin(.01)$ and $\cos(.01)$? How does this relate to the Small Angle Approximation?
6. Compute the quadratic approximation of e^x centered at 0. Estimate $e^{1/10}$ and $e = e^1$.
7. Compute the quadratic approximation of $\ln(1+x)$ centered at zero. Use this to estimate $\ln(1.1)$ and $\ln(2)$. How accurate do you expect these approximations to be? Check the true answers in Mathematica. Now try approximating $\ln(0)$.
8. If $f(x) = e^{x+x^2}$, find a formula for the quadratic approximation near zero, and use that to estimate $f(-.1)$.
9. Compute the quadratic approximation of $(1+x)^\alpha$ centered at 0. Use this formula to estimate 2^{10}. Use it to estimate 1.1^{10}.

Bonus: Special Relativity

Many formulas in the theory of special relativity depend on a parameter

$$\gamma(v) = \frac{1}{\sqrt{1-(v/c)^2}}$$

where v is the velocity, and c is the speed of light.

(a) What is $\gamma(0)$?

(b) Compute formulas for the linear and quadratic approximations to $\gamma(v)$ centered at zero. These tell us what happens when v is small relative to the speed of light.

(c) You are probably familiar with the famous formula that $E = mc^2$. This formula is for "rest energy", and holds when $v = 0$. For a moving object, we can compute the kinetic energy at a given velocity with the formula

$$E(v) = mc^2\gamma(v).$$

What happens if we replace γ with the quadrtic approximation? Does this look familiar?

Bonus: Simple Pendulums

We often want to model the motion of a pendulum. An application of Newton's laws to the forces of gravity on a pendulum give the differential equation

$$\theta'' = -\frac{g}{L}\sin(\theta)$$

where g is acceleration due to gravity, L is the length of the pendulum, and θ is the angle the pendulum makes with the vertical as a function of time. This is a differential equation, and it's far too complicated to solve. You can see this by entering the following command:

```
DSolve[y''[t] == - Sin[y[t]], y[t], t]
```

However, if we assume that the angle θ is always small, then we can use our quadratic approximation of $\sin(x)$ to see that $\sin(\theta) \approx \theta$. Then our differential equation becomes

$$\theta'' = -\frac{g}{L}\theta.$$

This equation is just simple harmonic motion, and has the solution $A\cos(t\sqrt{g/L})+B\sin(t\sqrt{g/L})$. The pendulum will complete one swing every $2\pi\sqrt{L/g}$ seconds. Notice that the angle doesn't appear in this expression, which explains Galileo's discovery that the duration of a pendulum swing doesn't depend on the size of the arc.

But Galileo was slightly wrong! This derivation depends on the approximation that $\sin(\theta) \approx \theta$, which holds when θ is small. If the pendulum is making a *large* swing, then we cannot use this approximation, and have to rely on the earlier, messy formula. The period does depend on the size of the arc; but if the arc is small, it doesn't depend on that much.

You can see an animation of the difference at `https://www.acs.psu.edu/drussell/Demos/Pendulum/Pendulum.html`.