Problem 1. Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a)

$$\lim_{x\to 9} \frac{3-\sqrt{x}}{9-x}$$

Solution:

$$\lim_{x\to 9} \frac{3-\sqrt{x}}{9-x} = \lim_{x\to 9} \frac{(3-\sqrt{x})(3+\sqrt{x})}{(9-x)(3+\sqrt{x})} = \lim_{x\to 9} \frac{9-x}{(9-x)(3+\sqrt{x})} = \lim_{x\to 9} \frac{1}{3+\sqrt{x}} = 1/6.$$

(b)

$$\lim_{x\to -\infty} \frac{3x^3+\sqrt[3]{x}}{\sqrt{9x^6+2x^2+1}+x}$$

Solution:

$$\begin{aligned}\lim_{x\to -\infty} \frac{3x^3+\sqrt[3]{x}}{\sqrt{9x^6+2x^2+1}+x} &= \lim_{x\to -\infty} \frac{3x^3/x^3+\sqrt[3]{x}/x^3}{\sqrt{9x^6+2x^2+1}/(-\sqrt{x^6})+x/x^3} \\ &= \lim_{x\to -\infty} \frac{3+x^{-8/3}}{-\sqrt{9+2x^{-4}+x^{-6}}+x^{-2}} \\ &= \lim_{x\to -\infty} \frac{3}{-\sqrt{9}} = -1.\end{aligned}$$

(c)

$$\lim_{x\to 1} \frac{\sin^2(x-1)}{(x-1)^2} =$$

Solution:

$$\lim_{x\to 1} \frac{\sin^2(x-1)}{(x-1)^2} = \lim_{x\to 1} \left(\frac{\sin(x-1)}{x-1}\right)^2 = \left(\lim_{x\to 1} \frac{\sin(x-1)}{x-1}\right)^2 = 1^2 = 1$$

by the small angle approximation.

(d)

$$\lim_{x\to 3} \frac{x-5}{(x-3)^2} =$$

Solution:

$$\lim_{x\to 3} \frac{x-5}{(x-3)^2} = -\infty$$

since the top approaches -2 and the bottom approaches zero and is always positive.

Problem 2.

(a) Using the Squeeze Theorem, show that

$$\lim_{x\to 3} \frac{x-3}{1+\sin^2\left(\frac{2\pi+e+7}{x-3}\right)} = 0.$$

Solution: Observe that since $-1 \le \sin(a) \le 1$ for any a, we have that $0 \le \sin^2(a) \le 1$ for any a, and thus $1 \le 1 + \sin^2(a) \le 2$. Taking the reciprocal gives us $1/2 \le \frac{1}{1+\sin^2(a)} \le 1$ for any a, and in particular for $a = \frac{2\pi+e+7}{x-3}$. We ignore the 1/2 and just look at the 1, since the 1 is larger, and get

$$-|x-3| \le \frac{x-3}{1+\sin^2\left(\frac{2\pi+e+7}{x-3}\right)} \le |x-3|.$$

By continuity, we can compute that $\lim_{x\to 3} |(x-3)| = \lim_{x\to 3} |x-3| = 0$. So by the squeeze theorem we know that

$$\lim_{x\to 3} \frac{x-3}{1+\sin^2\left(\frac{2\pi+e+7}{x-3}\right)} = 0.$$

(b) Let

$$g(x) = \begin{cases} \frac{x^2-1}{x-1} & x > 0 \\ x^2+1 & x < 0 \end{cases}$$

If possible, define an extension of g that is continuous at all real numbers.

Solution: g fails to be defined at 2 points: 0 and 1. We see that

$$\lim_{x\to 1} g(x) = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} x+1 = 2$$

so we wish to set $g_F(1) = 2$. (Alternatively, we can just replace the $\frac{x^2-1}{x-1}$ with an $x+1$).

At 0, we need to compute the two one-sided limits. We have

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} x^2+1 = 1$$
$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \frac{x^2-1}{x-1} = \frac{-1}{-1} = 1.$$

Thus the discontinuity is removeable, and we want to set $g_F(0) = 1$. Thus our continuous extension is

$$g_F(x) = \begin{cases} x+1 & x > 0 \\ 1 & x = 0 \\ x^2+1 & x < 0 \end{cases} = \begin{cases} x+1 & x \ge 0 \\ x^2+1 & x \le 0 \end{cases}$$

Problem 3.

(a) **Directly from the definition of derivative,** compute the derivative of $f(x) = x^2 + \sqrt{x}$ at $a = 2$.

Solution:

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0} \frac{(2+h)^2 + \sqrt{2+h} - 2^2 - \sqrt{2}}{h} \\ &= \left(\lim_{h\to 0} \frac{4h+h^2}{h}\right) + \left(\lim_{h\to 0} \frac{(\sqrt{2+h}-\sqrt{2})(\sqrt{2+h}+\sqrt{2})}{h(\sqrt{2+h}+\sqrt{2})}\right) \\ &= \left(\lim_{h\to 0} 4+h\right) + \left(\lim_{h\to 0} \frac{1}{\sqrt{2+h}+\sqrt{2}}\right) \\ &= 4 + \frac{1}{2\sqrt{2}}. \end{aligned}$$

(b) **Naming each derivative rule used explicitly**, compute the derivative of $g(x) = x\sqrt{x^2+1}$.

Solution:

$$\begin{aligned}
g'(x) &= (x') \cdot \sqrt{x^2+1} + x \cdot \frac{d}{dx}\sqrt{x^2+1} && \text{product rule} \\
&= 1 \cdot \sqrt{x^2+1} + x \cdot \frac{d}{dx}\sqrt{x^2+1} && \text{identity} \\
&= \sqrt{x^2+1} + x\frac{1}{2}(x^2+1)^{-1/2}(x^2+1)' && \text{chain rule and power rule} \\
&= \sqrt{x^2+1} + \frac{x}{2}(x^2+1)^{-1/2}\left((x^2)' + 1'\right) && \text{Additivity} \\
&= \sqrt{x^2+1} + \frac{x}{2}(x^2+1)^{-1/2}(2x+0) && \text{power rule and constants}
\end{aligned}$$

Problem 4.

(a) Find an equation of the line tangent to $y = \frac{x^2-1}{x^2+1}$ at the point $(0, -1)$.

Solution: We have that

$$y' = \frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2}$$

so when $x = 0$ we have $y' = (0-0)/1 = 0$. The equation for a tangent line is $y = m(x - x_0) + y_0$, so the tangent line to this function at $(0, 1)$ is $y = 0(x-0) + (-1)$, or $y = -1$.

(b) Use a linear approximation to estimate $\sqrt{4.01}$.

Solution: Use $f(x) = \sqrt{x}$ and $a = 4$. Then we have

$$\begin{aligned}
f(4.01) &\approx f'(4)(x-4) + f(4) \\
&= \frac{1}{4}(.01) + 2 = 2.0025.
\end{aligned}$$

(c) Give equation for the linear approximation of the function $f(x) = x\sin(x)$ near the point $a = \pi/2$.

Solution: We calculate that $f(x) = \pi/2\sin(\pi/2) = \pi/2$, and $f'(x) = \sin(x) + x\cos(x)$, so $f'(\pi/2) = \sin(\pi/2) + \pi/2\cos(\pi/2) = 1$. So

$$f(x) \approx \pi/2 + 1(x - \pi/2) = x.$$

Problem 5. Compute the derivatives of the following functions using methods we have learned in class. Show enough work to justify your answers.

(a) $f(x) = \sec\left(\frac{\sqrt{x^2+1}}{x+2}\right)$

Solution:

$$f'(x) = \sec\left(\frac{\sqrt{x^2+1}}{x+2}\right) \cdot \tan\left(\frac{\sqrt{x^2+1}}{x+2}\right) \cdot \frac{\frac{1}{2}(x^2+1)^{-1/2}2x(x+2) - \sqrt{x^2+1}}{(x+2)^2}$$

(b) $g(x) = \sqrt[4]{\frac{x^3+\cos(x^2)}{\sin(x^3)+1}}$

Solution:

$$g'(x) = \frac{1}{4}\left(\frac{x^3+\cos(x^2)}{\sin(x^3)+1}\right)^{-3/4} \cdot \frac{(3x^2 - \sin(x^2)2x)(\sin(x^3)+1) - \cos(x^3)3x^2(x^3+\cos(x^2))}{(\sin(x^3)+1)^2}$$