Problem 1.

(a) Let $F(x) = 1/x + 1$ be the amount of pressure exerted on a beam in pounds per square inch at a point x inches to the right of its left end.

  (i) What does the derivative $F'(x)$ represent, and what are its units?

  Solution: The derivative $F'(x)$ is the rate at which pressure is increasing as you move to the right along the stick. Its units are pounds per square inch per inch, or pounds per cubic inch.

  (ii) Compute $F'(5)$. What does this tell you?

  Solution: $F'(x) = -1/x^2$ so $F'(5) = -1/25$. This means that if we are five inches to the right of the endpoint, moving one more inch to the right should decrease the pressure by about 1/25 of a pound per square inch.

(b) Suppose that $Q(p) = 3p^2 + 10p - 100$ is the number of widgets you can buy at a price of p dollars.

  (i) What does the derivative $Q'(p)$ represent, and what are its units?

  Solution: The derivative is the rate at which increasing the price increases the number of widgets you can buy (called the marginal elasticity of demand, though you don't need to know that on the test). Its units are widgets per dollar.

  (ii) Calculate $Q'(10)$. What does this tell you?

  Solution: $Q'(p) = 6p + 10$ so $Q'(10) = 70$. This means that if you are buying widgets for \$10, you can get approximately seventy more widgets if you raise your price ot \$11.

(c) A radioactive substance begins decaying from 100g of material. When it reaches 10g, it is decaying at rate of 1g per year. After how many years does this occur?

Solution: If $S(t)$ is the amount of substance in year t, then we have $S(t) = Ce^{rt}$, and thus $S(0) = 100 = C$. We know that $S'(t) = rCe^{rt} = rS(t)$, so when $S(t) = 10$ we have $-1 = r10$ and thus $r = -1/10$. This gives us $S(t) = 100e^{-t/10}$. Now we can solve $10 = 100e^{-t/10}$, which implies $10^{-1} = e^{-t/10}$ and thus $-\ln(10) = -t/10$. Thus $t = 10\ln(10) \approx 23$ years.

Problem 2.

(a) Suppose a 100-liter tank contains a mixture of salt and water. Salt is added to the tank at a constant rate, and leaves the tank at a rate proportional to the concentration of salt in the tank. Write a differential equation to model this scenario, and explain in a couple sentences what your variables mean and why your equation is a good model.

Solution: Let $S(t)$ be the amount of salt in the tank, measured in kilograms, as a function of time measured in minutes. Then we have

$$S'(t) = K - k\frac{S(t)}{100\text{L}}.$$

Here S' is the net rate at which the amount of salt in the tank increases; K is the rate at which it enters, and the rate at which it leaves is proportional to the concentration, which is $S(t)/100\text{L}$.

Alternatively, we could let $C(t)$ be the concentration of salt, measured in kilograms per liter, as a function of time measured in minutes. Then we have

$$100\text{L}C'(t) = K - kC(t).$$

The left-hand side now is the change in concentration times the volume, to reflect the net change in salt amount; the right-hand side has K for the amount flowing in, and the amount flowing out is proportional to the concentration $C(t)$.

(Notice these are actually the same equation, since $C(t) = S(t)/100\text{L}$).

(b) Check whether the function $f(x) = xe^x$ is a solution to the differential equation $y'' = 2y' - y$. Justify your answer.

Solution:

$$\begin{aligned} f'(x) &= e^x + xe^x \\ f''(x) &= e^x + e^x + xe^x = 2e^x + xe^x \\ 2f'(x) - f(x) &= 2e^x + 2xe^x - xe^x = 2e^x + xe^x = f''(x). \end{aligned}$$

(c) Compute $\frac{d}{dx}e^{\sin(x^3)+x}$.

Solution:

$$\frac{d}{dx}e^{\sin(x^3)+x} = e^{\sin(x^3)+x}\left(\cos(x^3)3x^2 + 1\right).$$

Problem 3.

(a) Suppose we have the differential equation $f'(t) = f(t)(1 - f(t)$, and $f(0) = 1/2$. Use Euler's method with three steps to approximate $f(3)$.

Solution:

$$\begin{aligned} f(1) &\approx f(0) + f'(0)(1-0) = \frac{1}{2} + \frac{1}{4}(1) = \frac{3}{4} \\ f(2) &\approx f(1) + f'(1)(2-1) \approx \frac{3}{4} + \frac{3}{4}\frac{1}{4} = \frac{15}{16} \\ f(3) &\approx f(2) + f'(2)(3-2) \approx \frac{15}{16} + \frac{15}{16}\frac{1}{16} = \frac{255}{256}. \end{aligned}$$

(b) Suppose we have the differential equation $f'(t) = f(t) - t$, with $f(1) = 2$. Use Euler's method with three steps to approximate $f(4)$.

Solution: We have

$$\begin{aligned} f(2) &\approx f'(1)(2-1) + f(1) = 1 + 2 = 3 \\ f(3) &\approx f'(2)(3-2) + f(2) \approx 1(3-2) + 3 = 4 \\ f(4) &\approx f'(3)(4-3) + f(3) \approx 1(4-3) + 4 = 5. \end{aligned}$$

Problem 4.

(a) Find a formula for y' in terms of x and y if $x^8 + x^4 + y^4 + y^6 = 1$.

Solution:

$$\begin{aligned} 8x^7 + 4x^3 + 4y^3\frac{dy}{dx} + 6y^5\frac{dy}{dx} &= 0 \\ 8x^7 + 4x^3 &= (4y^3 + 6y^5)\frac{dy}{dx} \\ -\frac{4x^7 + 2x^3}{2y^3 + 3y^5} &= \frac{dy}{dx}. \end{aligned}$$

(b) Find a tangent line to the curve given by $x^4 - 2x^2y^2 + y^4 = 16$ at the point $(\sqrt{5}, 1)$.

Solution: We use implicit differentiation, and find that

$$\begin{aligned} 4x^3 - 2\left((2xy^2 + x^2 2y\frac{dy}{dx}\right) + 4y^3\frac{dy}{dx} &= 0 \\ 4x^3 - 4xy^2 &= 4x^2y\frac{dy}{dx} - 4y^3\frac{dy}{dx} \\ \frac{4x^3 - 4xy^2}{4x^2y - 4y^3} &= \frac{dy}{dx} \end{aligned}$$

Thus at the point $(\sqrt{5}, 1)$ we have

$$\frac{dy}{dx} = \frac{4\sqrt{5}^3 - 4\sqrt{5} \cdot 1^2}{4\sqrt{5}^2 \cdot 1 - 4 \cdot 1^3} = \sqrt{5}\left(\frac{20-4}{20-4}\right) = \sqrt{5}.$$

Thus the equation of our tangent line is

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 1 &= \sqrt{5}(x - \sqrt{5}). \end{aligned}$$

(c) If $x^2 y = x + y$ find a formula for $\frac{d^2y}{dx^2}$ in terms of x and y.

Solution: We get

$$\begin{aligned} 2xy + x^2 y' &= 1 + y' \\ (x^2 - 1)y' &= 1 - 2xy \\ y' &= \frac{1 - 2xy}{x^2 - 1} \\ y'' &= \frac{-2(y + xy')(x^2 - 1) - 2x(1 - 2xy)}{(x^2 - 1)^2} \\ &= \frac{-2\left(y + x\frac{1-2xy}{x^2-1}\right)(x^2 - 1) - 2x(1 - 2xy)}{(x^2 - 1)^2} \end{aligned}$$

Problem 5.

(a) The surface area of a cube is given by the formula $A = 6s^2$ where s is the length of a side. If the side lengths are increasing by 2 inches per second, how fast is the surface area increasing when the area is 54 square inches?

Solution: We have the data $A = 6s^2, A = 54, s' = 2$. We take a derivative and see that $A' = 12ss'$, so we need to find s. But when $A = 54$ we have

$$\begin{aligned} 54 &= 6s^2 \\ 9 &= s^2 \\ 3 &= s \end{aligned}$$

and thus

$$A' = 12ss' = 12 \cdot 3 \cdot 2 = 72$$

so the area is increasing at 72 square inches per second.

(b) A car is driving down a road at 150 feet per second (this is about a hundred miles an hour). A camera is placed 200 feet from the road, which will rotate to follow and record the progress of the car. How quickly must the camera rotate when the car is fifty feet away from directly in front of the camera?

Solution: Let the car's position be x, and the angle at which the camera is pointing is θ. Then we have $x = 50, x' = 150$, and we are looking for θ'. We have the equation $\tan\theta = x/200$, and thus

$$\begin{aligned} \sec^2\theta \cdot \theta' &= \frac{x'}{200} \\ &= 150/200 = 3/4. \end{aligned}$$

But we know that our triangle has sides of length 50 and 200, so the hypotenuse must have length $\sqrt{50^2 + 200^2} = \sqrt{2500 + 40000} = \sqrt{42500} = 10\sqrt{425} = 50\sqrt{17}$. Thus $\sec\theta = \sqrt{17}/4$ and $\sec^2\theta = (\sqrt{17}/4)^2 = 17/16$, and we have

$$\begin{aligned} 17\theta'/16 &= 3/4 \\ \theta' &= 12/17 \approx .70588. \end{aligned}$$

Thus the camera must rotate at 12/17 radians/second.