Problem 1.

(a) Find $\lim_{x\to+\infty} \frac{(\ln(x))^2}{x}$.

Solution: Since $\lim_{x\to+\infty} \ln(x)^2 = \lim_{x\to+\infty} x = +\infty$ we can use L'Hôpital's Rule, and we have

$$\begin{aligned}\lim_{x\to+\infty} \frac{\ln(x)^2}{x} &= \lim_{x\to+\infty} \frac{2\ln(x)/x}{1} = \lim_{x\to+\infty} \frac{2\ln(x)}{x}\\ &= \lim_{x\to+\infty} \frac{2/x}{1} = \lim_{x\to+\infty} \frac{2}{x} = 0.\end{aligned}$$

(b) Find $\lim_{x\to 3} \frac{2x^3-9x^2+10x-3}{x^4-8x^3+16x^2-5x+6}$.

Solution: $\lim_{x\to 3} 2x^3-9x^2+10x-3 = \lim_{x\to 3} x^4-8x^3+16x^2-5x+6 = 0$, so we can use L'Hôpital's rule.

$$\begin{aligned}\lim_{x\to 3} \frac{2x^3-9x^2+10x-3}{x^4-8x^3+16x^2-5x+6} &= \lim_{x\to 3} \frac{6x^2-18x+10}{4x^3-24x^2+32x-5}\\ &= \frac{54-54+10}{108-216+96-5} = \frac{10}{-17}.\end{aligned}$$

(c) Find $\lim_{x\to 0} \frac{2\sin(x)-\sin(2x)}{x-\sin(x)}$.

Solution: $\lim_{x\to 0} 2\sin(x)-\sin(2x) = 0-0 = 0$, and $\lim_{x\to 0} x-\sin(x) = 0$, so we can use L'Hôpital's Rule.

$$\begin{aligned}\lim_{x\to 0} \frac{2\sin(x)-\sin(2x)}{x-\sin(x)} &= \lim_{x\to 0} \frac{2\cos(x)-2\cos(2x)}{1-\cos(x)} \to \frac{0}{0}\\ &= \lim_{x\to 0} \frac{-2\sin(x)+4\sin(2x)}{\sin(x)} \to \frac{0}{0}\\ &= \lim_{x\to 0} \frac{-2\cos(x)+8\cos(2x)}{\cos(x)} = \frac{6}{1} = 6.\end{aligned}$$

Problem 2.

(a) Compute $f'(x)$ where $f(x) = e^{\arctan(x^2)}$.

Solution:

$$f'(x) = e^{\arctan(x^2)}\left(\frac{1}{1+x^4}\right)2x$$

(b) Compute $g'(4)$ where $g(x) = \ln(x^3+3x+\sqrt{x})$.

Solution:

$$g'(x) = \frac{1}{x^3+3x+\sqrt{x}}\left(3x^2+3+\frac{1}{2\sqrt{x}}\right)$$

so

$$g'(4) = \frac{1}{4^3+3\cdot 4+\sqrt{4}}\left(3(4^2)+3+\frac{1}{2\sqrt{4}}\right) = \frac{51+\frac{1}{4}}{78} = \frac{205}{312}.$$

(c) Find the tangent line to $h(x) = \arcsin(e^x)$ at $\ln(1/2)$.

Solution: We have $h'(x) = \frac{1}{\sqrt{1-e^{2x}}} \cdot e^x$, so $h'(\ln(1/2)) = \frac{e^{\ln 1/2}}{\sqrt{1-e^{2\ln(1/2)}}} = \frac{1/2}{\sqrt{1-1/4}} = \frac{1}{\sqrt{3}}$. We also have $h(\ln(1/2)) = \arcsin(1/2) = \pi/6$.

Thus the equation of the tangent line is

$$y - \pi/6 = \frac{1}{\sqrt{3}}(x - \ln(1/2)).$$

Problem 3.

(a) Find the derivative of $f(x) = \cos(x)^x$.

Solution:

We have

$$\begin{aligned} \ln y &= x \ln(\cos(x)) \\ y'/y &= \ln(\cos(x)) + x\frac{-\sin(x)}{\cos(x)} \\ y' &= \cos(x)^x \left(\ln(\cos(x)) - x \tan x\right) \end{aligned}$$

(b) Let $j(x) = \sqrt[3]{x^5 + x^4 + x^3 + x^2 + 2x}$. Find $(j^{-1})'(4)$.

Solution: Plugging in numbers, we see that $j(2) = \sqrt[3]{32 + 16 + 8 + 4 + 4} = \sqrt[3]{64} = 4$. Then by the Inverse Function Theorem we have $(j^{-1})'(4) = \frac{1}{j'(2)}$. But

$$\begin{aligned} j'(x) &= \frac{1}{3}\left(x^5 + x^4 + x^3 + x^2 + 2x\right)^{-2/3}\left(5x^4 + 4x^3 + 3x^2 + 2x + 2\right) \\ j'(2) &= \frac{1}{3}(64)^{-2/3}\left(80 + 32 + 12 + 4 + 2\right) = \frac{130}{48} = \frac{65}{24}. \end{aligned}$$

Thus by the inverse function theorem we have

$$(j^{-1})'(4) = \frac{24}{65}.$$

(c) Compute the following. In all cases your answers should be exact, with no decimals, and no logs or exponentials or trig functions..

$$\ln(e^3) + \ln(3) + \ln(e/3) =$$

Solution: $3 + \ln(3 \cdot e/3) = 3 + 1 = 4$

$$\arcsin(-\sqrt{2}/2) =$$

Solution: $-\pi/4$

$$\cos(\arcsin(3/7) =$$

Solution: $\frac{\sqrt{49-9}}{7} = \frac{2\sqrt{10}}{7}$

Problem 4.

(a) Show that the polynomial $x^4 - 6x - 2$ has two real roots, that is, there are two (different!) real numbers a and b such that $a^4 - 6a - 2 = b^4 - 6b - 2 = 0$.

Solution: Set $f(x) = x^4 - 6x - 2$; since this is a polynomial function it must be continuous. We compute:

$$f(0) = -2 \qquad f(-1) = 5$$
$$f(1) = -7 \qquad f(2) = 2$$

We have $-2 < 0 < 5$, so by the Intermediate Value Theorem there is some a between -1 and 0 with $f(a) = 0$. Similarly, we have $-7 < 0 < 2$ so by the Intermediate Value theorem there is some b between 1 and 2 with $f(b) = 0$. Clearly a and b are different since $a < 0$ and $b > 1$, so a and b are two distinct roots to the polynomial $x^4 - 6x - 2$.

(b) Find the general form of an antiderivative for $3x^2 + \cos(x)$.

Solution: $x^3 + \sin(x) + C$.

(c) Find y' if $e^y + \ln(y) = x^2 + 1$.

Solution:

$$e^y \cdot y' + \frac{y'}{y} = 2x$$
$$y'(e^y + \frac{1}{y}) = 2x$$
$$y' = \frac{2x}{e^y + \frac{1}{y}}.$$

Problem 5.

(a) Use two iterations of Newton's Method starting at 2 to estimate $\sqrt[3]{7}$.

Solution:

We start with $x_0 = 2$. We need to take $f(x) = x^3 - 7$, so $f'(x) = 3x^2$. Then

$$x_1 = 2 - \frac{1}{12} = \frac{23}{12}$$
$$x_2 = \frac{23}{12} - \frac{71/1728}{529/48} = \frac{18215}{9522} \approx 1.91294$$

(b) Find the formula for the quadratic approximation of $g(x) = x^x$ near 1.

Solution: We have

$$g(x) = e^{x\ln(x)} \qquad g(1) = 1$$
$$g'(x) = e^{x\ln(x)}(\ln(x) + 1) \qquad g'(1) = 1$$
$$g''(x) = e^{x\ln(x)}(\ln(x) + 1)^2 + e^{x\ln(x)}(1/x) \qquad g''(1) = 1 + 1 = 2$$

and thus we have

$$g(x) \approx 1 + (x - 1) + \frac{2}{2}(x - 1)^2.$$

(c) Let $F(2) = 1$ and let $F'(x) = \frac{x+1}{x-1}$. Use three steps of numerical integration/modified Euler's method to estimate $F(5)$.

Solution:

$$F(3) \approx F(2) + F'(2)(3 - 2) = 1 + 3 = 4$$
$$F(4) \approx 4 + 2(1) = 6$$
$$F(5) \approx 6 + 5/3 = 23/3 \approx 7.67.$$