Math 114 Test 1 Solutions

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Problem 1. Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a) $\lim_{x \to 1^+} \frac{x+2}{x-1} =$

Solution:

$$\lim_{x \to 1^+} \frac{x+2}{x-1} = +\infty$$

because the top approaches 3, and the bottom approaches 0 from the right.

(b) $\lim_{x \to -1} \frac{x^2-1}{x+1} =$

Solution:

$$\lim_{x \to -1} \frac{x^2-1}{x+1} = \lim_{x \to -1} x - 1 = -2.$$

(c) $\lim_{x \to +\infty} \frac{\sqrt{x^4+x}}{3x^2-1} =$

Solution:

$$\lim_{x \to +\infty} \frac{\sqrt{x^4+x}}{3x^2-1} = \lim_{x \to +\infty} \frac{\sqrt{1+1/x^2}}{3-1/x^2} = 1/3.$$

(d) $\lim_{x \to 0} \frac{\sin(2x)}{\sin(3x)} =$

Solution:

$$\lim_{x \to 0} \frac{\sin(2x)}{\sin(3x)} = \lim_{x \to 0} \frac{\sin(2x)}{2x} \frac{3x}{\sin(3x)} \frac{3x}{2x} = \lim_{x \to 0} 1 \cdot 1 \cdot \frac{3}{2} = 3/2.$$

Problem 2.

(a) Using the Squeeze Theorem, show that

$$\lim_{x \to 2} (x-2)^2 \cos\left(\frac{x+2}{x-2}\right) = 0.$$

Solution: Observe that since $-1 \leq \cos(a) \leq 1$ for any a, we have that $-1 \leq \cos\left(\frac{x+2}{x-2}\right) \leq 1$. Multiplying by $(x-2)^2$ gives

$$-(x-2)^2 \leq (x-2)^2 \cos\left(\frac{x+2}{x-2}\right) \leq (x-2)^2.$$

Then we can compute that $\lim_{x \to 2} -(x-2)^2 = \lim_{x \to 2} (x-2)^2 = 0$, so by the squeeze theorem we know that

$$\lim_{x \to 2} (x-2)^2 \cos\left(\frac{x+2}{x-2}\right) = 0.$$

(b) Let

$$g(x) = \begin{cases} x^2 - 4 & x > -3 \\ 2 - x & x < -3 \end{cases}$$

If possible, define an extension of g that is continuous at all real numbers.

Solution: g fails to be defined at $x = -3$. So we compute the one-sided limits: we have

$$\lim_{x \to -3^-} g(x) = \lim_{x \to -3^-} 2 - x = 5$$
$$\lim_{x \to -3^+} g(x) = \lim_{x \to -3^+} x^2 - 4 = 9 - 4 = 5.$$

Thus the discontinuity is removeable, and we want to set $g_F(-3) = 5$. Thus our continuous extension is

$$g_F(x) = \begin{cases} x^2 - 4 & x > -3 \\ 5 & x = -3 \\ 2 - x & x < -3 \end{cases} = \begin{cases} x^2 - 4 & x \geq -3 \\ 2 - x & x \leq -3 \end{cases}$$

Problem 3.

(a) **Directly from the definition of derivative,** compute the derivative of $f(x) = \sqrt{x+3}$ at $a = 1$.

Solution:

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{4+h-4}}{h(\sqrt{4+h} + \sqrt{4})} \\ &= \lim_{h \to 0} \frac{h}{h(\sqrt{4+h} + 2)} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{4+h} + 2} = \frac{1}{4}. \end{aligned}$$

(b) **Naming each derivative rule used explicitly**, compute the derivative of $g(x) = \frac{x^2}{x\cos(x)}$.

Solution:

$$\begin{aligned} g'(x) &= \frac{(x^2)' x\cos(x) - x^2(x\cos(x))'}{x^2\cos^2(x)} && \text{Quotient Rule} \\ &= \frac{2x \cdot x\cos(x) - x^2(x\cos(x))'}{x^2\cos^2(x)} && \text{Power Rule} \\ &= \frac{2x^2\cos(x) - x^2(x' \cdot \cos(x) + x \cdot (\cos(x))')}{x^2\cos^2(x)} && \text{Product Rule} \\ &= \frac{2x^2\cos(x) - x^2(\cos(x) + x \cdot (\cos(x))')}{x^2\cos^2(x)} && \text{Identity} \\ &= \frac{2x^2\cos(x) - x^2(\cos(x) + x(-\sin(x)))}{x^2\cos^2(x)} && \text{Trigonometry} \end{aligned}$$

Problem 4.

(a) Find an equation of the line tangent to $y = x\sqrt{x^3+1}$ at the point $(2, 6)$.

Solution: We have

$$y' = \sqrt{x^3+1} + x\frac{1}{2}(x^3+1)^{-1/2} \cdot 3x^2$$
$$y'(2) = 3 + 2 \cdot 12\frac{1}{3} \cdot 12 = 7.$$

so the equation of the tangent line is

$$y - 6 = 7(x-2)$$

or

$$y = 6 + 7(x-2) = 7x - 8.$$

(b) Use a linear approximation to estimate $\sqrt[3]{8.3}$.

Solution: Use $f(x) = \sqrt[3]{x}$ and $a = 8$. Then we have $f'(x) = \frac{1}{3}x^{-2/3}$ and $f'(a) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$, and

$$f(x)f(8) + \approx f'(8)(x-8)$$
$$f(8.3) \approx 2 + \frac{1}{12}(8.3-8) = 2 + \frac{1}{40} = \frac{81}{40}.$$

(c) Give an equation for the linear approximation of the function $f(x) = x^2\cos(x-1)$ near the point $a = 1$.

Solution: We calculate that $f(1) = 1$, and

$$f'(x) = 2x\cos(x-1) + x^2(-\sin(x-1))$$
$$f'(1) = 2 + 0 = 2$$
$$f(x) \approx 1 + 2(x-1) = 2x - 1.$$

Problem 5. Compute the derivatives of the following functions using methods we have learned in class. Show enough work to justify your answers.

(a) $f(x) = \dfrac{\cos^3(x^2+1)}{\sqrt{\tan(x)}}$

Solution:

$$\frac{d}{dx}f(x) = \frac{3\cos^2(x^2+1)(-\sin(x^2+1))2x\sqrt{\tan(x)} - \frac{1}{2}(\tan(x))^{-1/2}\sec^2(x)\cos^3(x^2+1)}{\tan(x)}$$

(b) $g(x) = \csc^3\left(\sec(x^2)\tan(x^4+1)\right)$

Solution:

$$\frac{d}{dx}g(x) = 3\csc^2\left(\sec(x^2)\tan(x^4+1)\right)\left(-\csc\left(\sec(x^2)\tan(x^4+1)\right)\cot\left(\sec(x^2)\tan(x^4+1)\right)\right)$$
$$\cdot\left(\sec(x^2)\tan(x^2)2x\tan(x^4+1) + \sec^2(x^4+1)4x^3\sec(x^2)\right)$$