Week 8: Elliptic Curve Cryptography

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Occidental College

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$\mathbb{F}_{13}$

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$$7^2 \equiv 10 \qquad 8^2 \equiv 12 \qquad 9^2 \equiv 3 \qquad 10^2 \equiv 9 \qquad 11^2 \equiv 4 \qquad 12^2 \equiv 1$$

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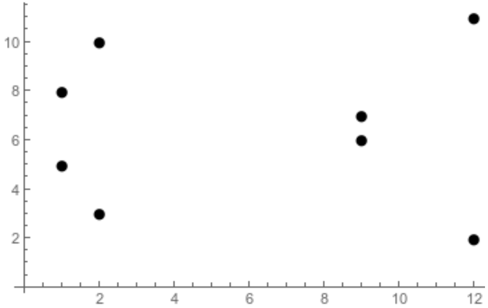
$$E(\mathbb{F}_{13}) = \{\mathcal{O}, (1,5), (1,8), (2,3), (2,10), (9,6), (9,7), (12,2), (12,11)\}.$$



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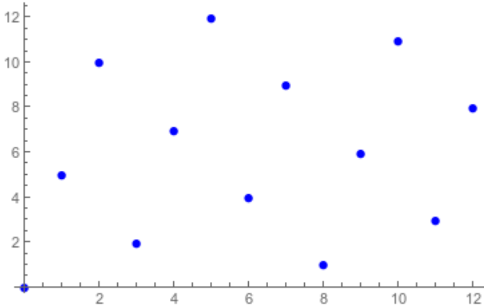
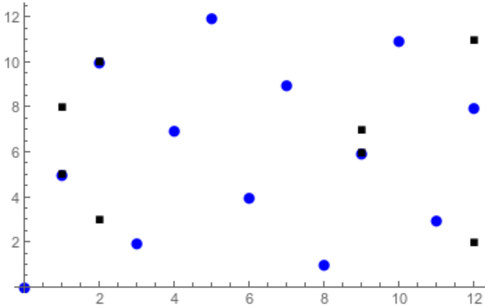


The line $y = 5x$ over $\mathbb{F}_{13}$



$$y^2 = x^3 + 3x + 8 \text{ and } y = 5x$$

Proposition

Let $E : y^2 = x^3 + Ax + B$ be an elliptic curve over $\mathbb{Q}$, and let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be points on $E(\mathbb{Q})$. Then:

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2. *If* $P_1 = P_2$*, then define* $\lambda = \frac{3x_1^2 + A}{2y_1}$*. Set*

$$x_3 = \lambda^2 - x_1 - x_2 \qquad y_3 = \lambda(x_1 - x_3) - y_1.$$

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3. *If* $P_1 \neq P_2$*, then define* $\lambda = \frac{y_2 - y_1}{x_2 - x_1}$*. Then as before, set*

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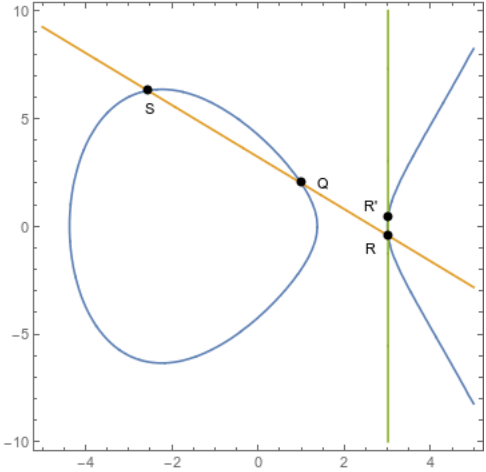
10
5
0
−5
−10
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3. Now we can compute

$$nP = (c_0 + c_1 \cdot 2 + c_2 \cdot 2^2 + \cdots + c_k 2^k)P = c_0 P \oplus c_1 2P \oplus c_2 4P \oplus \cdots \oplus$$

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4. Now Alice computes $n_A Q_B$ and Bob computes $n_B Q_A$.

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2. Bob computes $C_1 = kP \in E({}_p), C_2 = M + kQ_A \in E({}_p)$. Bob transmits the pair of points (C_1, C_2) to Alice.

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1. Alice computes $C_2 - n_A C_1 \in E({}_p)$.

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Symmetric Key Size	RSA Key Size	ECC Key Size
80	1024	160
112	2048	224
128	3072	256
192	7680	384
256	15360	521