Week 7: Elliptic Curves

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Occidental College

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It is possible to write endlessly on elliptic curves.

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Serge Lang

Groups

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Definition

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3. *Associative: for every $f, g, h \in G$ we have $(f \star g) \star h = f \star (g \star h)$.*

An example of the infinite dihedral group. We can accomplish any symmetry by combining a translation of some number of units with a possible 180° rotation.

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1. *K is an abelian group under the operation $+$;*
2. *The set $K \setminus \{0\}$ of non-zero elements of K is an abelian group under $\cdot$;*
3. *and we have the distributive law $k(x+y) = kx + ky$.*

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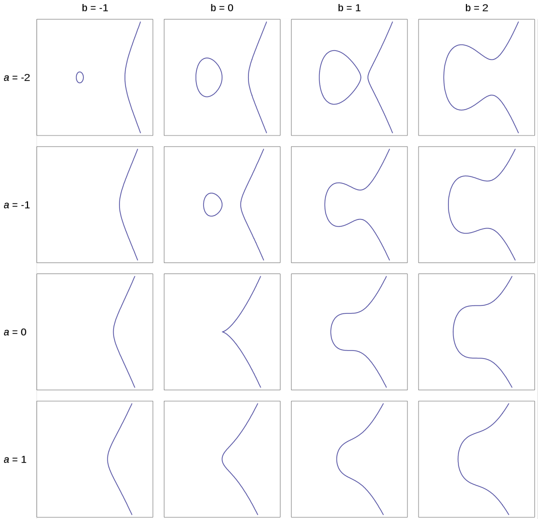

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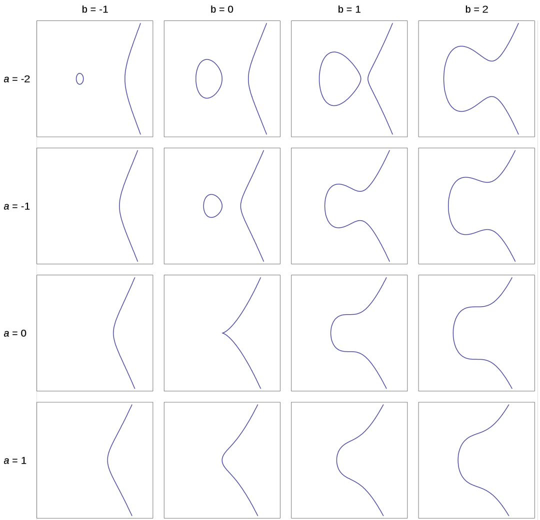
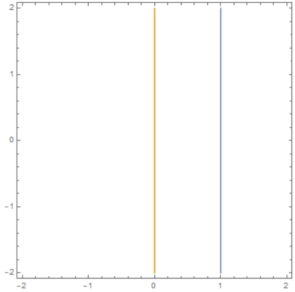
Key Question

How many rational points are there?

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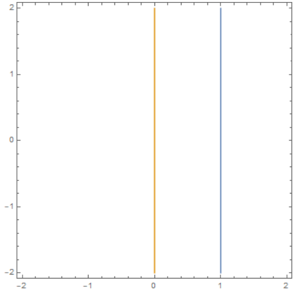
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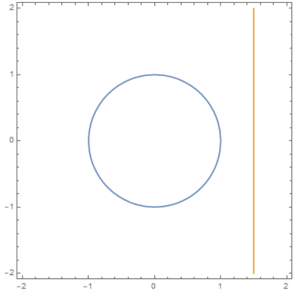
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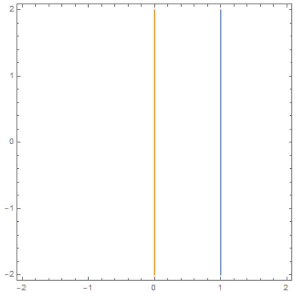
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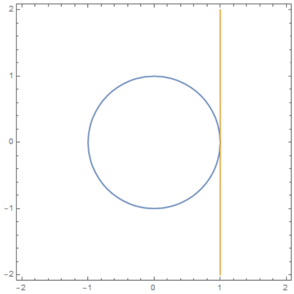
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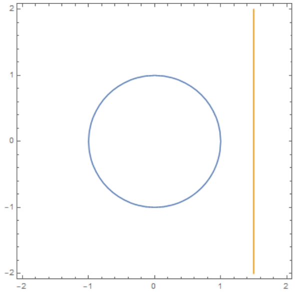
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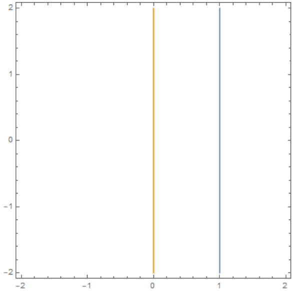
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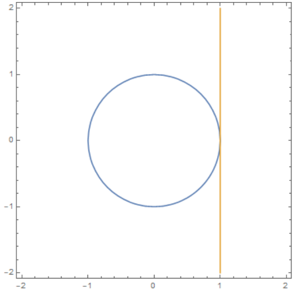
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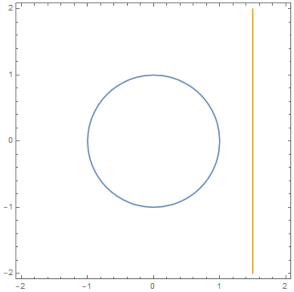
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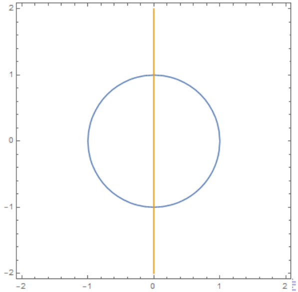
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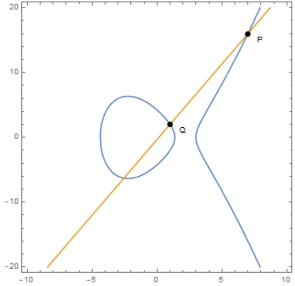
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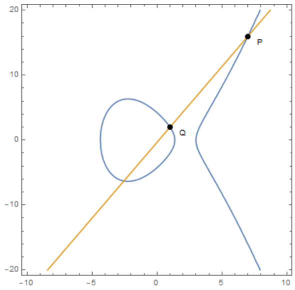
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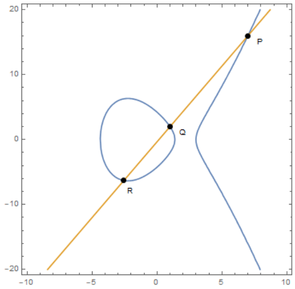
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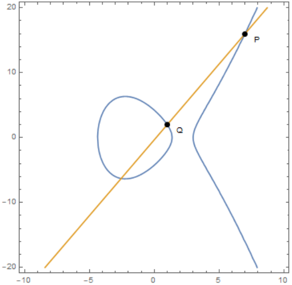
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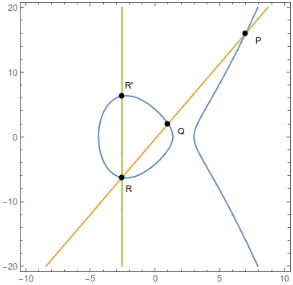
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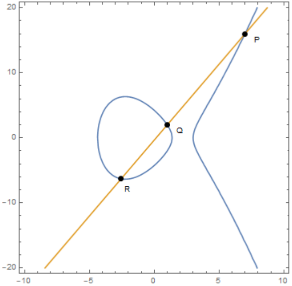
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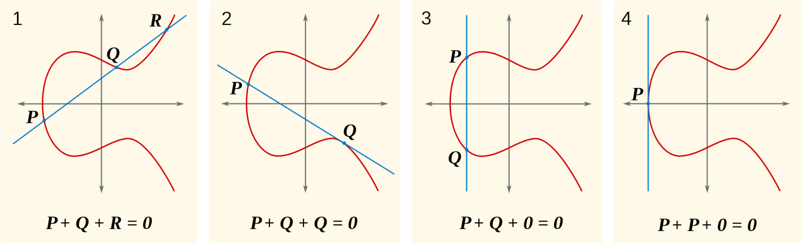
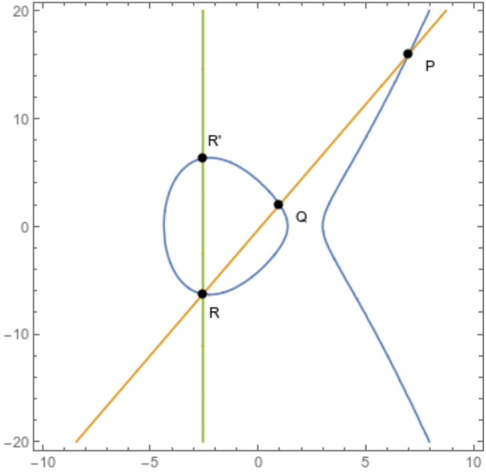


Figure: The group law on elliptic curves
Emmanuel Boutet / CC-BY-SA-3.0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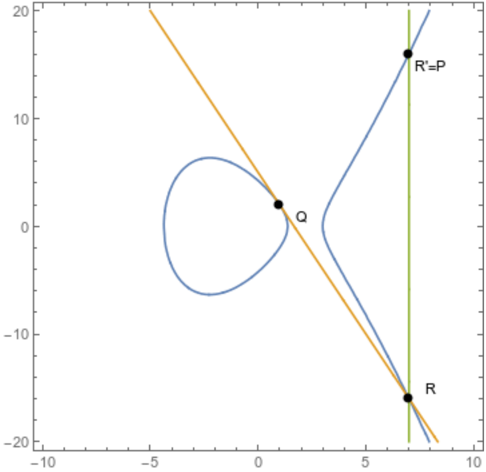
R'=P
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