

# Math 401 Fall 2019

## Cryptology HW 7 Solutions

### Due Thursday, October 17

Problems 2 and 3 will be worth 20 points each.

1. The group  $S_3$  is the set  $\{e, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau\}$ , where  $e$  is the identity and multiplication obeys the following rules:  $\sigma^3 = e = \tau^2, \tau\sigma = \sigma^2\tau$ .
  - (a) What are  $\sigma^{-1}$  and  $\tau^{-1}$ ? That is, tell me which of the six elements in the set I gave you is  $\sigma^{-1}$  and which is  $\tau^{-1}$ .
  - (b) Compute  $\tau\sigma^2, \tau(\sigma\tau), (\sigma\tau)(\sigma\tau)$ , and  $(\sigma\tau)(\sigma^2\tau)$ . (Again, your answer for each part should be one of the six elements I gave you.)

**Solution:**

- (a)  $\sigma^{-1} = \sigma^2$  and  $\tau^{-1} = \tau$ .
- (b)

$$\begin{aligned} \tau\sigma^2 &= \sigma^2\tau\sigma = \sigma^4\tau = \sigma\tau \\ \tau\sigma\tau &= \sigma^2\tau\tau = \sigma^2 \\ \sigma\tau\sigma\tau &= \sigma\sigma^2\tau\tau = \sigma^3\tau^2 = ee = e \\ \sigma\tau\sigma^2\tau &= \sigma\sigma\tau\tau = \sigma^2. \end{aligned}$$

2. Let  $E : y^2 = x^3 - 2x + 4$ , and let  $P = (0, 2)$  and  $Q = (3, -5)$ .
  - (a) Check that  $P, Q \in E(\mathbb{Q})$ .
  - (b) Compute  $\Delta$  to confirm that this is an elliptic curve.
  - (c) Compute  $P \oplus Q$ .
  - (d) Compute  $P \oplus P$  and  $Q \oplus Q$ .
  - (e) Compute  $3P = P \oplus P \oplus P$  and  $3Q = Q \oplus Q \oplus Q$ .

**Solution:**

- (a)  $0^3 - 2 \cdot 0 + 4 = 4 = 2^2$  and  $3^3 - 2 \cdot 3 + 4 = 25 = 5^2$  so these are on the curve.
- (b)  $\Delta = 4A^3 + 27B^2 = 4 \cdot (-2)^3 + 27 \cdot 4^2 = -32 + 432 = 400 \neq 0$  so this is an elliptic curve.

(c) The line through  $P$  and  $Q$  is given by  $y = \frac{-5-2}{3-0}(x-0) + 2 = -\frac{7}{3}x + 2$ .

We plug this into the cubic and get

$$\begin{aligned}(2 - 7x/3)^2 &= x^3 - 2x + 4 \\ 4 - 28x/3 + 49x^2/9 &= x^3 - 2x + 4 \\ 0 &= x^3 - 49x^2/9 + 22x/3\end{aligned}$$

and we know that  $-49/9 = -x_1 - x_2 - x_3 = -0 - 3 - x_3$  so  $-22/9 = x_3$  and  $x_3 = 22/9$ .

Plugging this back into the equation for the line, we get  $y_3 = -154/27 + 2 = -100/27$ , so  $\mathbf{P} \oplus \mathbf{Q} = (22/9, 100/27)$ .

(d) We have  $2yy' = 3x^2 - 2$ , so at the point  $P$  we have  $2 \cdot 2 \cdot y' = 3 \cdot 0^2 - 2$  or  $y' = -1/2$ . Thus the tangent line is  $y = -x/2 + 2$ .

Plugging this into the cubic gives

$$\begin{aligned}(-x/2 + 2)^2 &= x^3 - 2x + 4 \\ x^2/4 - 2x + 4 &= x^3 - 2x + 4 \\ 0 &= x^3 - x^2/4\end{aligned}$$

and we have  $-1/4 = -x_1 - x_2 - x_3 = -0 - 0 - x_3$  so  $x_3 = 1/4$ .

Plugging this into the line gives  $y_3 = -1/8 + 2 = 15/8$ . Thus we have  $P \oplus P = (1/4, 15/8)$ .

Similarly, at the point  $Q$  we have  $2 \cdot (-5) \cdot y' = 3 \cdot 3^2 - 2$  or  $-10y' = 25$ , so  $y' = -5/2$ . Thus the tangent line is  $y = -5/2(x-3) - 5 = -5x/2 + 5/2$ .

Plugging this into the cubic gives

$$\begin{aligned}(-5x/2 + 5/2)^2 &= x^3 - 2x + 4 \\ 25x^2/4 - 25x/2 + 25/4 &= x^3 - 2x + 4 \\ 0 &= x^3 - 25x^2/4 + 21x/2 - 9/4\end{aligned}$$

and we have  $-25/4 = -x_1 - x_2 - x_3 = -3 - 3 - x_3$  so  $-1/4 = -x_3$  and  $x_3 = 1/4$ .

Plugging this into the line gives  $y_3 = -5/8 + 5/2 = 15/8$ , so  $\mathbf{Q} \oplus \mathbf{Q} = (1/4, 15/8)$ .

(e) We have  $3\mathbf{Q} = (-237/121, -845/1331)$  and  $3\mathbf{P} = (240, 3718)$ .

In particular,  $3P = (1/4, 15/8) \oplus (0, 2)$ . The line connecting these two points is  $y = \frac{2+15/8}{0-1/4}(x-0) + 2 = -31x/2 + 2$ .

Plugging this into the cubic gives

$$\begin{aligned}(2 - 31x/2)^2 &= x^3 - 2x + 4 \\ 4 - 62x + 961x^2/4 &= x^3 - 2x + 4 \\ 0 &= x^3 - 961x^2/4 + 60x\end{aligned}$$

and we have  $-961/4 = -x_1 - x_2 - x_3 = -0 - 1/4 - x_3$  so  $x_3 = 960/4 = 240$ .

Plugging this back into the line gives  $y_3 = -31(240)/2 + 2 = -3718$ , so  $3P = (240, 3718)$ .

Similarly,  $3Q = (1/4, -15/8) \oplus (3, -5)$ . The line connecting these two points is  $y = \frac{-5+15/8}{3-1/4}(x-3) - 5 = \frac{-25/8}{11/4}(x-3) - 5 = \frac{-25}{22}(x-3) - 5 = \frac{-25}{22}x - \frac{35}{22}$ .

Plugging this into the cubic gives

$$\begin{aligned} (-25x/22 - 35/22)^2 &= x^3 - 2x + 4 \\ 1225/484 + 875x/242 + 625x^2/484 &= x^3 - 2x + 4 \\ 0 &= x^3 - 625x^2/484 - 1359x/242 + 711/484 \end{aligned}$$

and we have  $-625/484 = -x_1 - x_2 - x_3 = -3 - 1/4 - x_3$  so  $x_3 = -237/121$ .

Plugging this back into the line gives  $y_3 = (-25/22)(-237/121) - 35/22 = 845/1331$ . So  $3Q = (-237/121, -845/1331)$ .

3. Let  $E : y^2 = x^3 + 17$ . Let  $P = (-1, 4)$  and let  $Q = (2, 5)$ .

- Confirm that  $P, Q \in E(\mathbb{Q})$ .
- Compute  $\Delta$  to confirm that this is an elliptic curve.
- Compute  $P \oplus Q$  and  $P - Q$ .
- Compute  $2P = P \oplus P$  and  $2Q = Q \oplus Q$ .

**Solution:**

- $(-1)^3 + 17 = 15 = 4^2$  so  $P \in E(\mathbb{Q})$ , and  $2^3 + 17 = 25 = 5^2$  so  $Q \in E(\mathbb{Q})$ .
- $\Delta = 4A^3 + 27B^2 = 27 \cdot 17^2 = 7803 \neq 0$  so this is an elliptic curve.
- The line through  $P$  and  $Q$  is given by  $y = \frac{5-4}{2+1}(x+1) + 4 = x/3 + 13/3$ .

Plugging this into the cubic gives

$$\begin{aligned} (x/3 + 13/3)^2 &= x^3 + 17 \\ x^2/9 + 26x/9 + 169/9 &= x^3 + 17 \\ 0 &= x^3 - x^2/9 + -26x/9 - 16/9 \end{aligned}$$

so we have  $-1/9 = -x_1 - x_2 - x_3 = 1 - 2 - x_3$  and thus  $x_3 = -8/9$ .

Plugging this back into the line gives us  $y_3 = -8/27 + 13/3 = 109/27$ , so  $\mathbf{P} \oplus \mathbf{Q} = (-8/9, -109/27)$ .

$-Q = (2, -5)$ , so the line through  $P$  and  $-Q$  is  $y = \frac{-5-4}{2+1}(x+1) + 4 = -3x + 1$ .

Plugging this into the cubic gives

$$\begin{aligned} (1 - 3x)^2 &= x^3 + 17 \\ 1 - 6x + 9x^2 &= x^3 + 17 \\ 0 &= x^3 - 9x^2 + 6x + 16 \end{aligned}$$

so we have  $-9 = -x_1 - x_2 - x_3 = 1 - 2 - x_3$  and thus  $x_3 = 8$ .

Plugging this back into the line gives us  $y_3 = -24 + 1 = -23$ , so  $\mathbf{P} - \mathbf{Q} = (8, 23)$ .

- The derivative of the cubic is  $2yy' = 3x^2$ . So the derivative at  $P$  is  $2 \cdot 4 \cdot y' = 3(-1)^2$  so we have  $y' = 3/8$  and the equation for the tangent line is  $y = 3/8(x+1) + 4 = 3x/8 + 35/8$ .

Plugging this into the cubic gives

$$\begin{aligned}(3x/8 + 35/8)^2 &= x^3 + 17 \\ 9x^2/64 + 105x/32 + 1225/64 &= x^3 + 17 \\ 0 &= x^3 - 9x^2/64 - 105x/32 - 137/64\end{aligned}$$

so we have  $-9/64 = -x_1 - x_2 - x_3 = 1 + 1 - x_3$  and so we have  $x_3 = 137/64$ .

Plugging this back into the line gives us  $y_3 = 3 \cdot 137/8 \cdot 64 + 35/8 = 2651/512$ , so  $\mathbf{2P} = (\mathbf{137/64}, \mathbf{-2651/512})$ .

The derivative at  $Q$  is given by  $2 \cdot 5y' = 3 \cdot 2^2$  and thus  $y' = 6/5$ . Thus the equation for the tangent line is  $y = 6/5(x - 2) + 5 = 6x/5 + 13/5$ .

Plugging this into the cubic gives

$$\begin{aligned}(6x/5 + 13/5)^2 &= x^3 + 17 \\ 36x^2/25 + 156x/25 + 169/25 &= x^3 + 17 \\ 0 &= x^3 - 36x^2/25 - 156x/25 + 256/25\end{aligned}$$

so we have  $36/25 = -2 - 2 - x_3$  and thus  $x_3 = -64/25$ .

Plugging this into the line gives  $y_3 = 6/5(-64/25) + 13/5 = -59/125$  so  $\mathbf{2Q} = (\mathbf{-64/25}, \mathbf{59/125})$ .

4. Consider the following curves:

- (i)  $y^2 = x^3 - 7x + 3$
- (ii)  $y^2 = x^3 - 7x + 9$
- (iii)  $y^2 = x^3 - 7x - 12$
- (iv)  $y^2 = x^3 - 3x + 2$
- (v)  $y^2 = x^3$ .

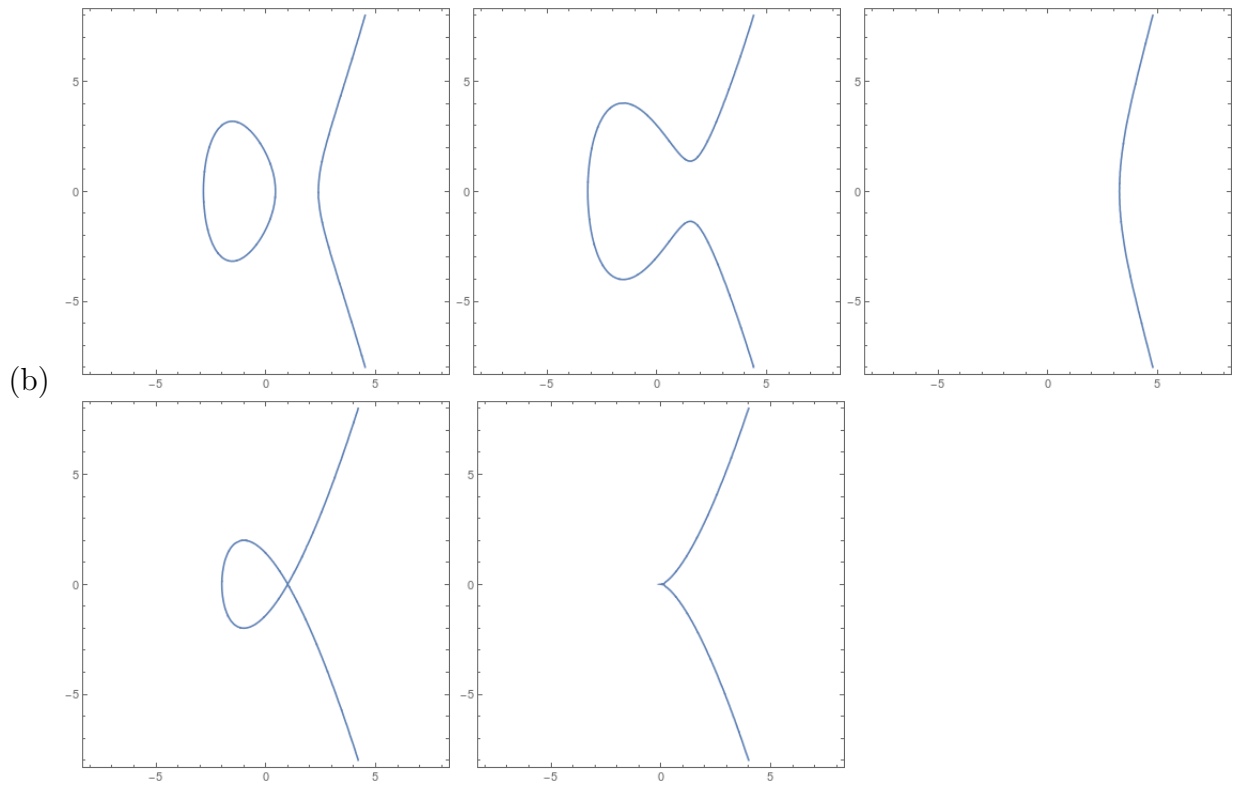
- (a) Compute the discriminant of each curve. Which of these are elliptic curves?
- (b) Sketch a graph of each curve (you may use a computer for this step). How can you visually tell which of these curves was an elliptic curve?

**Solution:**

- (a) The discriminants are:

- (i)  $-1129$
- (ii)  $815$
- (iii)  $2516$
- (iv)  $0$
- (v)  $0$

So the first three are elliptic curves, and the last two are not.



We see the first three are smooth curves, and thus are reasonable elliptic curves. The fourth has a self-intersection and the fifth has a cusp, so are not differentiable everywhere, and so the elliptic curve addition isn't always well-defined.