Lab 10 Tuesday April 23

Quadratic Approximation

Exercises

For each exercise, plot the graphs of the true function, the linear approximation, and the quadratic approximation. Makes sure they match!

1. Use a quadratic approximation to estimate $\sqrt[3]{28}$.

   Solution: Take $f(x) = \sqrt[3]{x}$ and $a = 27$. We have $f'(x) = \frac{1}{3}x^{-2/3}$ and $f''(x) = \frac{-2}{9}x^{-5/3}$ and thus

$$\begin{aligned} f'(27) &= \frac{1}{27} \\ f''(27) &= \frac{-2}{9} \cdot \frac{1}{243} = \frac{-2}{2187} \\ f(x) &\approx f(27) + f'(27)(x-27) + \frac{f''(27)}{2}(x-27)^2 \\ &= 3 + \frac{1}{27}(x-27) - \frac{1}{2187}(x-27)^2 \\ f(28) &\approx 3 + \frac{1}{27} - \frac{1}{2187} \approx 3.03658 \end{aligned}$$

2. If $f(x) = (x+8)^{1/2}$, compute a quadratic approximation centered at $x = 1$. Use this to estimate $f(1.02) = \sqrt{9.02}$.

   Solution:

$$\begin{aligned} f'(x) &= \frac{1}{2}(x+8)^{-1/2} \\ f'(1) &= \frac{1}{6} \\ f''(x) &= \frac{-1}{4}(x+8)^{-3/2} \\ f''(1) &= \frac{-1}{4 \cdot 27} = \frac{-1}{108} \\ f(x) &\approx 3 + \frac{1}{6}(x-1) - \frac{1}{54}(x-1)^2 \\ f(1.02) &\approx 3 + \frac{.02}{6} - \frac{(.02)^2}{108} = 3 + \frac{1}{300} + \frac{1}{270000} \\ &= \frac{20273}{6750} \approx 3.00333. \end{aligned}$$

3. If $f(x) = x^2 - 2x - 3$, compute a quadratic approximation centered at $x = 3$. How does this approximation compare to your original function?

Solution:

$$\begin{aligned} f'(x) &= 2x - 2 \\ f'(3) &= 4 \\ f''(x) &= 2 \\ f''(3) &= 2 \\ f(x) &\approx 0 + 4(x-3) + (x-3)^2 \\ &= 4x - 12 + x^2 - 6x + 9 = x^2 - 2x - 3 = f(x). \end{aligned}$$

This "approximation" is perfect, since our original equation was just a quadratic. We can use this general technique to rewrite polynomials in terms of $(x - a)$ for any a that will be useful to us.

4. Let $g(x) = x^4 - 3x^3 + 4x^2 + 4x - 2$. Compute the quadratic approximation at $x = -2$. Compare that to $g(x)$. Use this to estimate $g(-1.97)$.

   Solution:

$$\begin{aligned} g(2) &= 16 + 24 + 16 - 8 - 2 = 46 \\ g'(x) &= 4x^3 - 9x^2 + 8x + 4 \\ g'(-2) &= -32 - 24 - 16 + 4 = -80 \\ g''(x) &= 12x^2 - 18x + 8 \\ g''(-2) &= 48 + 36 + 8 = 92 \\ g(x) &\approx 46 - 80(x+2) + 46(x+2)^2 \\ f(-1.97) &\approx 46 - 80(.03) + 46(.009) = 43.6414. \end{aligned}$$

5. Compute the quadratic approximations of $\sin(x)$ and $\cos(x)$ centered at zero. Estimate $\sin(.01)$ and $\cos(.01)$? How does this relate to the Small Angle Approximation?

   Solution:

$$\begin{aligned} \sin'(x) &= \cos(x) \\ \sin'(0) &= 1 \\ \sin''(x) &= -\sin(x) \\ \sin''(0) &= 0 \\ \sin(x) &\approx 0 + 1(x-0) + \frac{0}{2}(x-0)^2 = x \\ \sin(.01) &\approx .01. \end{aligned}$$

$$\begin{aligned}
\cos'(x) &= -\sin(x) \\
\cos'(0) &= 0 \\
\cos''(x) &= -\cos(x) \\
\cos''(0) &= -1 \\
\cos(x) &\approx 1 + 0(x-0) - 1(x-0)^2 = 1 - \frac{x^2}{2} \\
\cos(.01) &\approx .99995.
\end{aligned}$$

6. Compute the quadratic approximation of e^x centered at 0. Estimate $e^{1/10}$ and $e = e^1$.

   Solution:

$$\begin{aligned}
f'(x) &= e^x \\
f'(0) &= 1 \\
f''(x) &= e^x \\
f''(0) &= 1 \\
e^x &\approx 1 + 1(x-0) + \frac{1}{2}(x-0)^2 = 1 + x + \frac{x^2}{2} \\
e^{.1} &\approx 1 + .1 + \frac{.01}{2} = 1.105 \\
e^1 &\approx 1 + 1 + \frac{1}{2} = 2.5.
\end{aligned}$$

7. Compute the quadratic approximation of $\ln(1+x)$ centered at zero. Use this to estimate $\ln(1.1)$ and $\ln(2)$. How accurate do you expect these approximations to be? Check the true answers in Mathematica. Now try approximating $\ln(0)$.

   Solution:

   $f'(x) = \frac{1}{1+x}$ and $f''(x) = \frac{-1}{(1+x)^2}$, so $f'(0) = 1$ and $f''(0) = -1$. Thus

$$\begin{aligned}
\ln(1+x) &\approx 0 + x - \frac{x^2}{2} \\
\ln(1.1) &\approx 0 + .1 - \frac{.01}{2} = .095 \\
\ln(2) &\approx 1 - \frac{1}{2} = .5 \\
\ln(0) &\approx 0 - 1 - \frac{1}{2} = -3/2.
\end{aligned}$$

   Since the true values are $\approx .0953$ and $\approx .7$, the first approximation is quite good, and the second is not too bad. Of course, we can do better either by using more steps or by taking a higher-order approximation.

   $\ln(0)$ is undefined, but $\lim_{x \to 0^+} \ln(0) = -\infty$. Thus our approximation $\ln(0) \approx -3/2$ is really bad. This is because all the third derivative gets really large near 0.

8. If $f(x) = e^{x+x^2}$, find a formula for the quadratic approximation near zero, and use that to estimate $f(-.1)$.

   Solution:

$$\begin{aligned} f(x) &= e^{x+x^2} & f(0) &= 1 \\ f'(x) &= e^{x+x^2}(1+2x) & f'(0) &= 1 \\ f''(x) &= e^{x+x^2}(1+2x)^2 + 2e^{x+x^2} & f''(0) &= 3 \end{aligned}$$

   so for our approximation we have

$$f(x) \approx 1 + 1(x-0) + \frac{3}{2}(x-0)^2 = 1 + x + 3x^2/2.$$

   Then we get

$$f(-.1) \approx 1 - .1 + .03/2 = .915.$$

   The true answer is approximately 9.13931.

9. Compute the quadratic approximation of $(1+x)^\alpha$ centered at 0. Use this formula to estimate 2^{10}. Use it to estimate 1.1^{10}.

   Solution:

$$\begin{aligned} f'(x) &= \alpha(1+x)^{\alpha-1} \\ f'(0) &= \alpha \\ f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2} \\ f''(0) &= \alpha(\alpha-1) \\ (1+x)^\alpha &\approx 1 + \alpha(x-0) + \frac{\alpha(\alpha-1)}{2}(x-0)^2 = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 \\ 2^{10} &\approx 1 + 10 \cdot 1 + \frac{10 \cdot 9}{2}1^2 = 1 + 10 + 45 = 56 \\ 1.1^{10} &\approx 1 + 10 \cdot .1 + 45 \cdot .01 = 2.45. \end{aligned}$$

Bonus: Special Relativity

Many formulas in the theory of special relativity depend on a parameter

$$\gamma(v) = \frac{1}{\sqrt{1-(v/c)^2}}$$

where v is the velocity, and c is the speed of light.

(a) What is $\gamma(0)$?

(b) Compute formulas for the linear and quadratic approximations to $\gamma(v)$ centered at zero. These tell us what happens when v is small relative to the speed of light.

(c) You are probably familiar with the famous formula that $E = mc^2$. This formula is for "rest energy", and holds when $v = 0$. For a moving object, we can compute the kinetic energy at a given velocity with the formula

$$E(v) = mc^2\gamma(v).$$

What happens if we replace γ with the quadrtic approximation? Does this look familiar?

Solution:

(a) $\gamma(0) = 1$.

(b)

$$\begin{aligned}
\gamma'(v) &= \frac{-1}{2}(1-(v/c)^2)^{-3/2} \cdot (-2v/c^2) \\
\gamma'(0) &= 0 \\
\gamma(v) &\approx \gamma(0) + 0(v-0) = 1
\end{aligned}$$

so the linear approximation isn't very helpful.

For the quadratic approximation we have

$$\begin{aligned}
\gamma''(v) &= \frac{3v^2}{c^4}(1-(v/c)^2)^{-5/2} + \frac{1}{c^2}(1-(v/c)^2)^{-3/2} \\
\gamma''(0) &= \frac{1}{c^2} \\
\gamma(v) &\approx \gamma(0) + 0(v-0) + \frac{1}{2c^2}(v-0)^2 = 1 + \frac{v^2}{2c^2}.
\end{aligned}$$

(You could also get this by doing a first-order "binomial" expansion on $f(x) = \sqrt{1+x}$ and then plugging in $x = v/c^2$).

(c) We get

$$E(v) = mc^2(1 + v^2/2c^2) = mc^2 + \frac{1}{2}mv^2.$$

This is rest energy plus kinetic energy, where kinetic energy is given by the usual high school physics formula. This tells you that kinetic energy is *approximately* equal to $\frac{1}{2}mv^2$. This approximation is very good at speeds that are small relative to the speed of light.