Lab 2 Tuesday January 29

Filling in functions

We can always take a function whose domain is not all reals and extend it to a function whose domain is all real numbers by simply assigning values to all the points that aren't already covered. You can think of this as defining a piecewise function where one of the pieces is just a single point. We often want to extend a given function to be a *continuous* function on all reals; but this isn't always possible.

Go to the course webpage and download the file called "plot piecewise file". If the computer asks to open it with Mathematica, do that; if not, copy all the text (use ctrl+a) and then paste it into an empty Mathematica notebook, and when asked if you want to interpret the text, click "yes". Click anywhere in the giant block of code and hit shift+enter; this gives you the new function `PlotPiecewise` which we will be using for the rest of this lab.

For each of the following functions, do the following steps:

1. Figure out the domain, and define the function `f[x]` in mathematica.
2. Plot the function using the `PlotPiecewise` command from the notebook I posted on Moodle, and the command `PlotPiecewise[f[x],{x,-3,3}]`
   Use the domain given in the problem.
3. Can you assign values to the function at the holes so that the function is "nice" or "connected" (continuous)? ("No" is a possible answer here, but think about why).
4. If you can, define a new function `fFixed[x]` that has the holes filled in, using the `Piecewise` command. E.g. if there is a hole at $x = 2$ and you think the value "should" be 5, enter `fFixed[x_]:= Piecewise[{{f[x],x!= 2},{5,x==2}}]`
   (note we use `!=` for "not equals" and `==` for "equals")
5. Plot your new function in `PlotPiecewise` to see if you have successfully filled the holes.
   `PlotPiecewise[fFixed[x],{x,-3,3}]`

(a) `f[x_]:= (x^2-1)/(x-1)` on $(-2, 2)$.

(b) `g[x_]:= 1/(x+1)` on $(-2, 2)$.

(c) `h[x_]:= (x+1)^2/(x^2-1)` on $(-2, 2)$.

(d) `i[x_]:= (x^2-2x+1)/(x-1)` on $(-2, 2)$.

(e) `j[x_]:= 2 Abs[x+1]/(2x+2)` on $(-2, 2)$.

(f) `m[x_]:= Sin[x-1]/(x-1)` on $(0, 2)$.

(g) `n[x_]:= (x^3-8)/(x^2-4)` on $(-4, 4)$.

(h) `o[x_]:= (x^2-x)/(x^3-3x^2+2x)` on$(-2, 4)$.

Trigonometry

1. Run the code `Plot[Sin[x],{x,-2Pi,2Pi}]` Where does this function look continuous? What about `Cos[x]` and `Tan[x]`?
2. Now define a function `f[x_]:=Sin[1/x]`. What is its value at 0? Enter `f[0]` into Mathematica and see what you get. Now plug in about ten numbers getting close to 0. What happens?
3. Plot a graph of `f[x]` with domain $[-1, 1]$. What happens near zero? Try using smaller domains like $[-.1, .1]$ or $[-.01, .01]$.
4. Run the code `Limit[f[x], x ->0]` to compute the limit at 0. Does this match what you see?

5. Define `g[x_]:=x * Sin[1/x]`. Before plotting it, what do you think will happen when x is near zero? Plug in some points near 0 to improve your guess.
6. Plot a graph of `g[x]` with domain $[-1, 1]$. What happens near zero? Plot more graphs with smaller domains. Then use the `Limit` command to compute the limit at 0.
7. Set `h[x_]:=Sin[x]/x`. Before plotting it, what do you expect to happen near zero? Try plugging in a few points near zero to improve your guess.
8. Plot a graph of `h[x]` with domain `[-Pi,Pi]`. What happens near zero? Plot more graphs with smaller domains. Also plot some larger domains. Do these pictures match your expectations?
9. Use the `Limit` command to compute the limit at 0.

The Squeeze Theorem

A principle called the "Squeeze Theorem" or the "Two Policemen Theorem" allows us to compute the limit of a function we don't like by "trapping" or "squeezing" it between two functions which do. In the rest of this lab we'll visualize a few examples. We'll discuss more in class soon.

1. Earlier in this lab, we considered the function `x*Sin[1/x]` and its limit at zero. Now we want to generate better understanding of its behavior there.
   (a) Plot the function $x \sin(1/x)$ with the code `Plot[x * Sin[1/x],{x,-1,1}]`.
   (b) Now plot $x \sin(1/x)$ on the same graph as $|x|$ and $-|x|$, with the code `Plot[{x * Sin[1/x], Abs[x],-Abs[x]},{x,-1,1}]`
   (c) Plot the function again with smaller domains centered at $x = 0$ to see what happens.
   (d) What are the limits of $|x|$ and $-|x|$ at 0? What does this suggest about the limit of $x \sin(1/x)$?
2. We also looked at the important limit `Sin[x]/x`.
   (a) Plot a graph of `Sin[x]/x`, along with the functions `1` and `Cos[x]`.
   (b) Shrink the domain around $x = 0$ to see what happens. What are the limits of 1 and of $\cos(x)$ at 0? What does this suggest about the limit of $\sin(x)/x$?
3. Now we'll practice finding these bounds. We'll study the function $(x + 1)\cos^2\left(\frac{32+x}{x+1}\right)$; we want to use the domain $[-2, 0]$. As a tip, in Mathematica you need to type `Cos[x]^2` to get $\cos^2(x)$.
   (a) Using the fact that $-1 \leq \cos(a) \leq 1$ for any a, find upper and lower bounds for $\cos\left(\frac{32+x}{x+1}\right)$. Find a number that's always bigger than this function, and another that's always smaller.
   (b) Use your answer to find bounds for $\cos^2\left(\frac{32+x}{x+1}\right)$. Again, you should find a number that's always bigger and one that's always smaller. (Plot this function to check your answer!)
   (c) Now find bounds for $(x + 1)\cos^2\left(\frac{32+x}{x+1}\right)$. You should have a *function* that's always bigger and a function that's always smaller.
   (d) Plot all three functions from the previous part with domain $[-2, 0]$. Is your upper bound actually always above the function? Is your lower bound always below? Make sure the bounds don't cross *through* the function. If they do, can you fix this?
   (e) What does this picture suggest about $\lim x \to -1(x + 1)\cos^2\left(\frac{32+x}{x+1}\right)$?
4. Use the same process for $(x + 1)^2\cos^2\left(\frac{32+x}{x+1}\right)$. What's very different about this graph?