Lab 4 Tuesday February 12

Secant Lines and Tangent Lines

In class we've been talking about the derivative as a way to approximate a function with a simpler, linear function. Now we want to talk about what this means graphically. (In a couple weeks we'll talk about how to interpret this physically).

First we recall some facts about lines. There are a few line equations we may want to use:

$$y = mx + b \qquad \text{Slope-Intercept Formula}$$
$$y - y_0 = m(x - x_0) \qquad \text{Point-Slope Formula}$$
$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) \qquad \text{Two Points Formula}$$

In order to write down the equation of a line, we need two pieces of information: either we need two points, or we need one point and also the slope. In class, we've used information about $f(a)$ to find the slope $f'(a)$.

But what if we know the values of $f(a)$ and also $f(b)$? Then we can use the two-points formula to write the equation of a line through those two points:

$$f(x) - f(b) = \frac{f(b) - f(a)}{b - a}(x - a).$$

And this is *almost* the linear approximation formula, since $f'(a) \approx \frac{f(b)-f(a)}{b-a}$. As b gets closer to a, this will get closer and closer to being the linear approximation formula.

So what does this mean geometrically? First we should talk about two types of lines from a geometric perspective.

Definition 0.1. A line that touches a curve at one point without crossing it is *tangent* to the curve at that point, and we call such a line a *tangent line* (from Latin *tangere* "to touch".)

A line crossing a curve in two points is called a *secant* line. (from Latin *secare* "to cut").

Just as the tangent of an angle is the length of a (specific) tangent line segment, the secant of an angle is the length of a (specific) secant line segment.

Given numbers a and b, a line through $(a, f(a))$ and $(b, f(b))$ is a secant line. As b gets closer to a, then the two points the secant line goes through get closer together. When we take the limit, our line "goes through the same point twice". Thus it only touches the curve at one point—so it is a tangent line. Thus we see that the linear approximation to a function at a point a is the line tangent at that point a.

And geometrically this should make sense. The tangent line touches the function graph at one point, and is going in the "same direction" as the graph at that point. Thus it's the line that looks most like the point. So it *should* be the line that best approximates that function.

Secant and Tangent Line Exercises

In this lab, we will see the process of approximating a curve via a secant line. At the bottom of this sheet, there is a list of functions and of x-values. **For each pair f, a you should do the following steps**; I show you code for the pair `f[x_] := x^3/2-x` and `a=0`.

1. Plot the graph of f, and imagine what the tangent like will look at near a.

   ```
   f[x_] := x^3/2-x
   Plot[f[x],{x,-2,2}]
   ```

2. Pick some points close to a, such as $a+1$, $a+.1$, $a-1$, $a-.1$, and so forth. On one graph, plot the function f, and the line connecting two points on the graph near a (using the two-point equation. The equation for this line in Mathematica code is `(x-a)(f[b]-f[a])/(b-a)+f[a]`).

   I pick the points $-1, -.5, -.1, -.01, 1, .5, .1, .01$; I'll show code for lines connecting the first two of these points to $(0, f(0)$ below.

   ```
   Plot[{f[x],(x-0) (f[-1]-f[0])/(-1-0) + f[0]},{x,-2,2}]
   Plot[{f[x],(x-0) (f[-.5]-f[0])/(-.5-0) + f[0]},{x,-2,2}]
   ```

   When you follow these steps for each function, make sure you **try some points on the right and some points on the left!** Usually the same thing will happen, but not always. What happens as the points get closer together?

3. Compute the slopes of each line you plotted. What is the limit as b gets closer to a?

4. Use Mathematica to compute the derivative at a, which should be the same as the limit you just computed. (Or compute it by hand!) There are two ways to do this; one of them is the obvious way. Try each at least once.

   `f'[0]` or `D[f[x],x]/.x->0`

5. Plot on the same graph f and the tangent line to f at a. Remember the point slope formula is y = m(x-a) + f(a).

```
Plot[{f[x], f'[0](x-0) + f[0]},{x,-1,1}]
```

Do all of the previous steps for each of the following functions and points a. For each problem after (c), it may be helpful to graph your functions with the `PlotRange` option:

```
Plot[{f[x], (f[b]-f[a])/(b-a)(x-a) + f[a]}, {x,a-1,a+1}, PlotRange->{-3,3}]
```

(a) `f[x_] := x^3/2 -x` and `a = 2`

(b) `g[x_] := x^5+1` and `a = 2`

(c) `h[x_] := x^3+x^2-1` and `a = -1`

(d) `Sin[x]` and `a = 0`

(e) `CubeRoot[x]` and `a=1`

(f) `CubeRoot[x]` and `a=0`

(g) `k[x_]:=Piecewise[{{-1,x<0},{1,x>=0}}]` and `a = 0`

(h) `m[x_]:=CubeRoot[x^2]` and `a = -1`

(i) `m[x_]:=CubeRoot[x^2]` and `a = 0`

(j) `Abs[x]` and `a = 0`