Lab 5 Tuesday February 19

Linear Approximation

We know that if we have a function $f(x)$ and know what it looks like at a point a, we can use the derivative to give a linear approximation

$$f(x) \approx f(a) + f'(x)(x-a).$$

Last lab, we drew secant lines, which are lines that intersect the graph of a function in (at least) two points; we may recall that by rearranging some information, we can write

$$f(x) = \frac{f(x)-f(a)}{x-a}(x-a) + f(a).$$

As x approaches a, this becomes closer to being a tangent line, and the slope term becomes closer to $f'(a)$. Thus we can get a decent approximation, if x and a are close, by replacing this difference quotient with the derivative:

$$f(x) \approx f'(a)(x-a) + f(a).$$

In this lab we want to push that idea a bit farther and see what we can do with it—and when it breaks down.

Error Margins

Now answer the following questions. In each problem, before you do any computations, think carefully about what you should use for f, a, b.

1. Estimate $(2.1)^5$ without doing any calculations. Then use the tangent line to approximate $(2.1)^5$. Finally, use Mathematica to compute the exact answer. How far off were you?
2. Now approximate $(2.5)^5$ using $a = 2$. Approximate 3^5 using $a = 2$. Are your approximations getting better or worse? Why? What does this tell you about what counts as "close" to 2?
3. Without calculating, find an upper bound and a lower bound for $(4.5)^3$. (Hint: $4 < 4.5 < 5$). Now approximate 4.5^3 with a tangent line in two different ways, from two different base points (that is, two different choices of a). What happens?
4. Approximate `CubeRoot[28]` and `82^(1/4)`.
5. Now approximate `28^3` and `82^4` using the same base points a that you used in the last problem. Are these approximations better or worse than your approximations of `CubeRoot[28]` and `82^(1/4)` above? Why? Would you do this on your own?
6. If you take $a = 0$ and $f(x) = x^{10}$, use a tangent line to approximate $f(2)$. What happens and why? What if you instead approximate with $a = 1$?

Finding Formulas

Sometimes rather than just approximating a specific number, we want to actually use a linear formula to approximate our function. For each of these problems you should find a formula for the linear approximation. Then graph your approximation and your original function together and see how they compare.

1. Approximate `Sin[.05]` and `Cos[.05]` Note that this is .05 and not .5.
2. Find a formula to approximate $\sin(x)$ when x is "small". (This is the revenge of the Small Angle Approximation). Find a formula to approximate $\cos(x)$ when x is small. What's unusual about this second formula?
3. Find formulas to approximate $\tan(x)$ and $\sec(x)$ near $a = 0$. How do these formulas relate to the ones you worked out for sin and cos?
4. Find formulas to approximate $\sin(x)$ and $\cos(x)$ near $a = \pi/2$. How do these relate to the formulas from number 2?
5. Find formulas to approximate $\sin(x)$ and $\cos(x)$ near $a = \pi/4$. What do you notice here?
6. Can you find formulas to linearly approximate $\cot(x)$ and $\csc(x)$ near $a = 0$?
7. Find a formula to approximate $f(x) = x^3 + 3x^2 + 5x + 1$ near $a = 0$. What do you notice? Why does that happen?
8. Find a formula to linearly approximate $f(x) = \frac{1}{1-x}$ near $x = 0$. (Hint: we can do this without the chain rule using the quotient rule).
9. Can we linearly approximate $f(x) = 1/x$ near 0?
10. Can we linearly approximate $f(x) = 1/x$ near 1?
11. Approximate $(1.01)^{10}$.
12. Approximate $(1.01)^\alpha$ where $\alpha \neq 0$ is some constant (your answer will have an α in it).
13. Now find a formula to approximate $(1 + x)^\alpha$ where x is "small" and $\alpha \neq 0$ is a constant. (This rule is the called the "binomial approximation" and is often useful in physics).
14. Bonus: Can you find a formula to approximate $(1 + x^n)^\alpha$ when x is small?