Lab 6 Tuesday, March 5

Euler's Method

Download the file `euler.nb` from the course web page, and evaluate the first block of code. This will give you four functions.

`euler[df,x0,y0,xfin,n]` uses Euler's method to approximate $f(x_{fin})$ given the initial data $y' = df, f(x_0) = y_0$ and n steps. It outputs the estimate of $f(x_{fin})$.

`eulerplot[df,x0,y0,xfin,n]` runs the same approximation, but instead of reporting the estimate as a number, it plots the data points.

`comparewitherrors[f_, df_, x0_, y0_, xfin_, n_]` takes a known function and plots that function on the domain $[x_0, x_{fin}]$; against it it plots the data points generated by using Euler's method to estimate $f(x_{fin})$ with the initial data $f(x_0) = y_0$.

1. Consider the example equation $f'(t) = f(t) - f(t)^2/2$ with $f(0) = 1$.
   (a) Use the command `euler[y - y^2/2, 0, 1, 3, 3]` to estimate $f(3)$ given $y' = y - y^2/2$ and $f(0) = 1$, with three steps.
   (b) Use the command `eulerplot[y - y^2/2, 0, 1, 3, 3]` to see these results graphically.
   (c) Now try using nine steps, with `euler[y - y^2/2,0,1,3,9]`. What changes? Do the same with `eulerplot` . Now try using 100 steps.
2. Now let's play around with the comparison plot and the function $f(x) = e^x$, which satisfies $y' = y$.
   (a) Use the function `comparewitherrors[E^x,y, 0,1,5,5]` to compare the actual function to the results of Euler's method.
   (b) Why did we pick the initial values we did?
   (c) Try again with 10 steps instead of 5. Try 100 steps. Play around and see what happens as you change the step size.
3. The differential equation $y' = y - e^x \sin(5x)/2 + 5e^x \cos(5x)$ with initial conditions $y(0) = 0$ has solution $y = e^x \sin(5)$.
   (a) Run `comparewitherrors[E^x Sin[5x],y-E^x Sin[5x]/2+5E^x Cos[5x],0,0,10,10]`
   (b) Try with 100 steps. Try with 1000.
4. If we take $y' = 2y/x - x^2y^2$, you can check that $f(x) = \frac{5x^2}{x^5+4}$ is a solution, with initial condition $f(1) = 1$.
   (a) Use the command `comparewitherrors[5 x^2/(x^5 + 4), 2 y/x - x^2 y^2, 1, 1, 5, 10]` to use Euler's method to fit the equation. What happens? How is this different from before?
   (b) Increase the number of steps to 100. To 1000.

Population Growth Models

Let's model population growth, assuming that that population growth is proportional to current population, unconstrained by resources. We assume that $P'(t) = kP(t)$, which that $P(t) = Ce^{kt}$. We have two constants here: C and k.

In 1960, global population was 3 billion, and in 1975 global population was 4 billion. It will be convenient to treat 1960 as year zero for this problem.

1. Use the initial value data that $P(0) = 3$ to determine the constant C. Can you use algebraic tools to solve for k as well?

   Solution: Since $P(0) = 3$ we know that $Ce^0 = 3$ so $C = 3$.

   We can say that $4 = P(15) = 3e^{15k}$, so $e^{15k} = 4/3$. Then we can take a logarithm of both sides to get $15k = \ln(4/3)$ or $t = \ln(4/3)/15$. But if you can't see that, don't worry about it; we'll cover logarithms soon. It's about .2.

2. Experiment with the `euler` command and try to figure out values for C and k. Note that we want the initial condition to be $P(0) = 3$ and we want $P(15)$ to come out to be 4. Make sure you use enough steps to get a good result!

   Solution: I get $C = 3$ and $k \approx .2$.

3. Use the constants you just found and use Euler's method to estimate the global population in 1990. Estimate global population in 2020.

   Solution: This model estimates that the population in 1990 should be $P(30) = 5.33$ billion, and the population in 2020 should be $P(60) = 9.5$ billion.

4. In 1990, global population was approximately 5.3 billion; population in 2020 is projected to be 7.6 billion. Compare these results with your model. What does that tell you?

   Solution: Our estimate for 1990 was actually pretty good, but our estimate for 2020 was much too high. So our model is eventually overshooting.

5. According to your model, what will the global population be in 2100?

   Solution: The model gives us about 44 billion people in 2100. This seems fairly unlikely.

6. Now instead of using the `euler` function to estimate the population in 1990 and in 2020, use the formula $P(t) = Ce^{kt}$ with the constants you found. How does this change things?

   Solution: If we used enough steps, not very much at all. With enough steps, Euler's method will be quite accurate here.

   If we didn't use enough steps, the Euler's method estimate will be a bit too low.

Our first model didn't give great results, so let's use a better one: *logistic growth*, otherwise known as growth in the presence of resource constraints. We assume that global population has some maximum M, and we have growth jointly proportional to the population $P(t)$ and to the remaining capacity $M - P(t)$. Thus we have the equation $P'(t) = kP(t)\left(1 - \frac{P(t)}{M}\right)$.

1. We have three constants here: k, M, and the initial condition C. So we need three data points to find all of them. Fortunately, we know that the population was 3 billion in 1960, 4 billion in 1975, and 5.3 billion in 1990. Experiment with the `euler` function to find constants C, k, and M that give us all three of these values.

Solution: We have an initial population of $C = 3$, and then the best fit I get is something like $k = .025$ and $M = 17$. But there are various answers that work pretty okay.

2. We actually have better population estimates than that. Find a better estimate of C, k, and M if the population was 3.04 billion in 1960, 4.08 billion in 1975, and 5.28 billion in 1990.

   Solution: Now I get something like $k = .028$ and $M = 12$, but there are various other answers you can get.

   This is surprisingly sensitive to the exact numbers we plug in. Notice how we've changed our estimate of the maximum population!

3. Based on the values you found in part (a), what will the population be in 2020? In 2100? What happens in the limit as t goes to infinity? Then use the values from part (b) instead.

   Solution:

   With the values from part (a) I get 8.40 billion in 2020 and 14.93 billion in 2100. In the infinite limit, we get a population of about 17 billion.

   With the values from part (b) I get 7.75 billion in 2020 and 11.33 billion in 2100. In the infinite limit, we get a population of about 12 billion.

4. The current best estimates of global population are 7.58 billion in 2020 and about 11.2 billion in 2100. What does this tell you about your model?

   Solution: The part (b) model works out *really well* for me. The part (a) one is kind of miscalibrated.

5. The general solution to a logistic growth equation is $P(t) = \frac{MCe^{kt}}{M + C(e^{kt} - 1)}$, which is kind of a mess. What equations can we get by plugging in our boundary value data? Can we solve those? Do we want to?

6. Take the constants you found in part (a) and (b) and plug them into the formula from part (e). Does the formula give you the same values for 1960, 1975, and 1990 that you should get?

7. Plot the graph of this formula. What does it look like?