Lab 9 Tuesday April 9

Newton's Method

In class we've been talking about solving equations: given f and y, find an x such that $f(x) = y$. This is the same thing as asking for $f^{-1}(y)$ when f is one-to-one. But suppose we can't find that solution, or at least can't find it easily. Can we at least approximate it?

To keep things simple, we'll assume that we want to solve $f(x) = 0$. (If not, we can just subtract our number y from both sides of the equation). If we know the value of f and of f' at a point x_1, then recall that by linear approximation we estimate that $f(x_2) = f(x_1) + f'(x_1)(x_2 - x_1)$. Since we want $f(x_2) = 0$, we set $f(x_2) = 0$ and solve this equation for x_2, and get

$$x_2 = x_1 - (f(x_1)/f'(x_1)).$$

In many conditions, we will get the result that x_2 is closer to being a root of f than x_1 is.

We can repeat this process to find x_3, x_4, etc., and ideally each will be a better estimate than the previous estimate was. A good rule of thumb for when to stop: if you want five decimal places of accuracy, you can stop when the nth step and the $n + 1$st step agree to five decimal places.

This method does have limitations. First, we have to start with a guess x_1 for our root x. Second, if $f'(x_1)$ is very close to zero, Newton's method will work poorly if it works at all, and we might have to pick a better guess. But it can be very useful for finding approximate solutions to equations.

Useful Mathematica note: recall that we can evaluate the *same* line of code repeatedly, and we can use this fact to save a great deal of typing here. If we have defined a function `f` and a starting point `a`, we can run the following line of code repeatedly:
`a = a - f[a]/f'[a]` and it will repeat the process over and over again; we can just keep hitting shift+enter until we have the precision we wish.

1. (a) Starting with $x_1 = 1$, use `f[x_] : = x^4 - 2` to estimate $\sqrt[4]{2}$ to four decimal places. (Notice the extra work we've done here: a fourth root of 2 is a solution to the equation $x^4 = 2$ and thus a root of $x^4 - 2$. In general we need to turn every problem into the question of finding a *root* of a function).

   Do two iterations by hand, and then shift to Mathematica. It might be helpful to start Mathematica off with `x=1.` (note the decimal point!) to force the software to give you decimal answers, which are more readable.

   Solution: We have $f'(x) = 4x^3$, so compute

$$\begin{aligned} x_2 &= 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-1}{4} = 5/4 \\ x_3 &= 5/4 - \frac{f(5/4)}{f'(5/4)} = \frac{5}{4} - \frac{(625/256) - (512/256)}{125/16} = \frac{5}{4} - \frac{113/256}{125/16} \\ &= \frac{5}{4} - \frac{113}{125 \cdot 16} = \frac{2387}{2000} \end{aligned}$$

Switching to a computer,

$$x_4 = \frac{2387}{2000} - \frac{f(2387/2000)}{f'(2387/2000)} \approx 1.18923$$
$$x_5 = 1.18923 - \frac{f(1.18923)}{f'(1.18923)} \approx 1.18921.$$

(b) Plot the tangent line corresponding to the first two steps you did in part (a).

(c) Repeat part (a) from the beginning starting from $x_1 = -1$, and again starting from $x_1 = 0$.

Solution: You should get -1.18921 in the first case, and in the second case you get a divide-by-zero error (literally–a "singular Jacobian" means a derivative equal to zero).

(d) In Mathematica, run the commands `FindRoot[x^4 ==2,  {x,1}]`, `FindRoot[x^4 == 2, {x,-1}]`, and `FindRoot[x^4 == 2, {x, 0}]`.

2. (a) Plot a graph of both `Cos[x]` and `x` with the command `Plot[{Cos[x],x},{x,-2Pi, 2Pi}]`. About where does it look ike the two functions intersect?

(b) Using your guess from part (a) as a starting point, use Newton's method to estimate a solution to $\cos(x) = x$ that is correct to six decimal places.

Solution: Note that we are not asking for a root of $\cos(x)$! We want to know when $\cos(x) = x$, that is, when $\cos(x) - x = 0$. So we want a root of $f(x) = \cos(x) - x$.

Starting with a guess of $x_1 = \pi/4$ (other guesses, such as 1, are also reasonable), we get

$$x_2 = \pi/4 - \frac{f(\pi/4)}{f'(\pi/4)} = \pi/4 - \frac{\sqrt{2}/2 - \pi/4}{-\sqrt{2}/2 - 1} \approx .739536$$
$$x_3 = .739536 - \frac{f(.739536)}{f'(.739536)} = .739085$$

(c) Run the command `FindRoot[Cos[x]==x, {x,a}]`, where a is your guess from part (a).

3. (a) Starting with $x_1 = 1$, estimate the root to `g[x_]=x^3-x-1` to four decimal places.

Solution: We have $g'(x) = 3x^2 - 1$, so compute

$$x_2 = 1 - \frac{-1}{2} = \frac{3}{2}$$
$$x_3 = \frac{3}{2} - \frac{27/8 - 3/2 - 1}{27/4 - 1} \approx 1.34783$$
$$x_4 = 1.34783 - \frac{g(1.34783)}{g'(1.34783)} \approx 1.3252$$
$$x_5 \approx 1.32472$$

(b) Do the same, starting with $x_1 = .6$.

Solution: This should take much longer.

(c) Do the same, starting with $x_1 = .57$.

Solution: This should take nearly a hundred iterations if it converges at all.

(d) Plot g from -2 to 2. Why were (a), (b), and (c) so different? Try plotting some tangent lines.

Solution: There is a local minimum at about .577. .6 is close to that but on the same side as the (only) root, so it converges but slowly. .57 is on the "wrong" side and gets "trapped" there for a long time.

4. (a) Starting with $x_1 = 1$, use three iterations of Newton's method to find a solution to `CubeRoot[x] == 0`. What happens?

Solution: Starting with 1 we have

$$x_2 = 1 - \frac{\sqrt[3]{1}}{(1/3)1^{-2/3}} = 1 - 3 = -2$$
$$x_3 = -2 - \frac{\sqrt[3]{-2}}{1/3(-2)^{-2/3}} = -2 + 6 = 4$$
$$x_4 = -8$$

and so on. The method never converges, and in fact each new iteration gets us further from the root (which is zero) than the previous one did.

(b) Plot a graph of `CubeRoot[x]`. Graphically, why did you get the result you did in part (a)? Plot the tangent lines that correspond to the approximations you calculated.

Solution: There is a vertical tangent line at the root, so Newton's method gets confused.

We can use Newton's Method to solve problems where we otherwise would have to guess and check.

5. Let $f(x) = x^5 + x^3 + x$.

(a) Use Newton's method to approximate $f^{-1}(2)$–i.e., to approximate a solution for $f(x) = 2$.

Solution: We want a solution to $f(x) = 2$, so set $f_2(x) = x^5 + x^3 + x - 2$, and then we're looking for a root of f_2. We have $f_2'(x) = 5x^4 + 3x^2 + 1$. Take $x_1 = 1$ since we know that $f(1) = 3$ is close to 2; then we have

$$x_2 = 1 - \frac{f_2(1)}{f_2'(1)} = 1 - \frac{1}{9} = \frac{8}{9}$$
$$x_3 = \frac{8}{9} - \frac{f_2(8/9)}{f_2'(8/9)} = \frac{8}{9} - \frac{(8/9)^5 + (8/9)^3 + 8/9 - 2}{5(8/9)^4 + 3(8/9)^2 + 1} \approx .866376$$
$$x_4 = .865583$$

(b) Use the Inverse Function Theorem to approximate $(f^{-1})'(2)$.

Solution: We know from part (a) that $f^{-1}(2) \approx .865583$. So

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1})(2)} \approx \frac{1}{f'(.865583)} \approx \frac{1}{6.05446} \approx .165168.$$

6. Let $g(x) = \sqrt{1 + x + x^2 + x^3}$.

(a) Use Newton's Method to approximate $g^{-1}(3)$.

Solution: Again, we want a root of $g(x) - 3$ so let $g_1(x) = \sqrt{1 + x + x^2 + x^3} - 3$. Then $g_1'(x) = \frac{1+2x+3x^2}{2\sqrt{1+x+x^2+x^3}}$.
We can take $x_1 = 0$. Then

$$\begin{aligned} x_2 &= 0 - \frac{g_1(0)}{g_1'(0)} = 0 - \frac{-2}{1/2} = 4 \\ x_3 &= 4 - \frac{g_1(4)}{g_1'(4)} = 4 - \frac{\sqrt{85} - 3}{57/2\sqrt{85}} = 4 - \frac{2\sqrt{85}}{57}\left(\sqrt{85} - 3\right) \approx 1.988 \\ x_4 &= x_3 - \frac{g_1(x_3)}{g'(x_3)} \approx 1.60. \end{aligned}$$

The true answer is approximately 1.578.

(b) Use the Inverse Function theorem to approximate $(g^{-1})'(3)$.

Solution: We have $g^{-1}(3) \approx 1.6$ from part (c), and thus

$$(g^{-1})'(1) = \frac{1}{g'(g^{-1}(1))} \frac{1}{g'(1.6)} \approx \frac{1}{1.95} \approx .51.$$

7. (a) Approximate $\sqrt[5]{20}$ to eight decimal places.
   (b) Find four real roots of $x^6 - x^5 - 6x^4 - x^2 + x + 10$ to eight places.
   (c) Show that $x^4 - 3x^3 + 5x^2 - 6$ has a root in (1,2) (hint: IVT), and approximate it to six decimal places.
   (d) Approximate $\log 3$.

Bonus: Does $3^{x^2} - 9^{4x} - 6$ have a root? (How many does it have?) Can you estimate them?

After thinking about this, check out this Quora discussion and see a few different ways to approach this. Jafar Mortadha's answer uses Newton's method; Alon Amit's uses the Intermediate Value Theorem. Prahar Mitra's says he uses "Taylor series", which I have mentioned but will not be covered until Calculus II; but in practice he's just using linear approximations like we have been for the past couple months. All three answers show how we can use different tools we've developed to approach tricky questions.

`https://www.quora.com/How-do-you-solve-3-x-2-9-4x-+6/answer/Richard-Muller-3`