Math 114 Test 1 Solutions

Instructor: Jay Daigle

Problem 1. Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a) $\lim_{x \to -2} \frac{x}{x+2} =$

Solution:

$$\lim_{x \to -2} \frac{x}{x+2} = \pm\infty$$

because the top goes to -2 and the bottom goes to 0.

(b) $\lim_{x \to +\infty} \frac{x+1}{\sqrt{4x^2+3x+1}} =$

Solution:

$$\lim_{x \to +\infty} \frac{x+1}{\sqrt{4x^2+3x+1}} = \lim_{x \to +\infty} \frac{1+1/x}{\sqrt{4+3/x+1/x^2}} = \frac{1}{2}.$$

(c) $\lim_{x \to 3} \frac{x^2-9}{x^2-8} =$

Solution:

$$\lim_{x \to 3} \frac{x^2-9}{x^2-8} = \frac{0}{1} = 0.$$

(d) $\lim_{x \to 1^+} f(x) =$ where

$$f(x) = \begin{cases} 3x+5 & x > 1 \\ x^2-3 & x < 1 \end{cases}$$

Solution:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 3x+5 = 8.$$

Problem 2.

(a) Using the Squeeze Theorem, show that

$$\lim_{x \to -1} (x+1) \sin\left(\frac{1}{x+1}\right) = 0.$$

Solution: Observe that since $-1 \leq \sin(a) \leq 1$ for any a, we have that $-1 \leq \sin\left(\frac{1}{x+1}\right) \leq 1$. Taking absolute values and multiplying by $|x+1|$ gives

$$-|x+1| \leq (x+1)\sin\left(\frac{1}{x+1}\right) \leq |x+1|$$

or

$$0 \leq \left|(x+1)\sin\left(\frac{1}{x+1}\right)\right| \leq |x+1|.$$

Then we can compute that $\lim_{x\to-1} -|x+1| = \lim_{x\to-1} |x+1| = 0$, so by the squeeze theorem we know that

$$\lim_{x\to-1}(x+1)\sin\left(\frac{1}{x+1}\right) = 0.$$

(b) Let

$$g(x) = \begin{cases} \sin(x-2) & x > 2 \\ x^2 - 4 & x < 2 \end{cases}$$

If possible, define an extension of g that is continuous at all real numbers.

Solution: g fails to be defined at $x = 2$. So we compute the one-sided limits: we have

$$\lim_{x\to2^-} g(x) = \lim_{x\to2^-} x^2 - 4 = 0$$
$$\lim_{x\to0^+} g(x) = \lim_{x\to2^+} \sin(x-2) = \sin(0) = 0.$$

Thus the discontinuity is removeable, and we want to set $g_F(2) = 0$. Thus our continuous extension is

$$g_F(x) = \begin{cases} \sin(x-2) & x > 2 \\ 0 & x = 2 \\ x^2 - 4 & x < 2 \end{cases} = \begin{cases} \sin(x-2) & x \geq 2 \\ x^2 - 4 & x \leq 2 \end{cases}$$

Problem 3.

(a) **Directly from the definition of derivative,** compute the derivative of $f(x) = \frac{1}{x+2}$ at $a = 3$.

Solution:

$$\begin{aligned} f'(3) &= \lim_{h\to0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h\to0} \frac{\frac{1}{3+h+2} - \frac{1}{5}}{h} \\ &= \lim_{h\to0} \frac{\frac{5-(5+h)}{5(5+h)}}{h} \\ &= \lim_{h\to0} \frac{-h}{5(5+h)h} \\ &= \lim_{h\to0} \frac{-1}{5(5+h)} = \frac{-1}{25}. \end{aligned}$$

(b) **Naming each derivative rule used explicitly**, compute the derivative of $g(x) = 4x^2 + x\sin(x)$.

Solution:

$$\begin{aligned} g'(x) &= (4x^2)' + (x\sin(x))' && \text{Additivity} \\ &= 4(x^2)' + (x\sin(x))' && \text{Scalar products} \\ &= 4 \cdot 2x + (x\sin(x))' && \text{Power Rule} \\ &= 8x + x(\sin(x))' + (x)'\sin(x) && \text{Product rule} \\ &= 8x + x\cos(x) + (x)'\sin(x) && \text{Trig Rules} \\ &= 8x + x\cos(x) + 1 \cdot \sin(x) && \text{Identity/Power Rule.} \end{aligned}$$

Problem 4.

(a) Find an equation of the line tangent to $y = \sin(x^2)$ at the point $(\sqrt{\pi}, 0)$.

Solution: We have

$$\begin{aligned} y' &= \cos(x^2) \cdot 2x \\ y'(\sqrt{\pi}) &= \cos(\pi) \cdot 2\sqrt{\pi} = -2\sqrt{\pi} \end{aligned}$$

so the equation of the tangent line is

$$y - 0 = 0 - 2\sqrt{\pi}(x - \sqrt{\pi})$$

or

$$y = 2\pi - 2\sqrt{\pi}x.$$

(b) Use a linear approximation to estimate 3.1^4.

Solution: Use $f(x) = x^4$ and $a = 3$. Then we have

$$\begin{aligned} f(3.1) &\approx f'(3)(x-3) + f(3) \\ &= 108(.1) + 81 = 91.8. \end{aligned}$$

(c) Give an equation for the linear approximation of the function $f(x) = \frac{x}{x^2+1}$ near the point $a = 2$.

Solution: We calculate that $f(1) = \frac{2}{5}$, and

$$\begin{aligned} f'(x) &= \frac{(x^2+1) - 2x(x)}{(x^2+1)^2} \\ f'(1) &= \frac{4+1-8}{5^2} = \frac{-3}{25} \\ f(x) &\approx \frac{2}{5} - \frac{3}{25}(x-2). = \frac{16}{25} - \frac{3}{25}x \end{aligned}$$

Problem 5. Compute the derivatives of the following functions using methods we have learned in class. Show enough work to justify your answers.

(a) $f(x) = \dfrac{\sec(x^2+1)}{\tan(\sqrt{x})}$.

Solution:

$$\frac{d}{dx}f(x) = \frac{\sec(x^2+1)\tan(x^2+1)2x\tan(\sqrt{x}) - \sec^2(\sqrt{x})\frac{1}{2\sqrt{x}}\sec(x^2+1)}{\tan^2(\sqrt{x})}$$

(b) $g(x) = \sqrt[3]{\csc\left(\sin\left(x^3+x\right)\right)}$

Solution:

$$\frac{d}{dx}g(x) = \frac{1}{3}(\csc(\sin(x^3+x)))^{-2/3}(-\csc(\sin(x^3+x))\cot(\sin(x^3+x)))\cos(x^3+x)(3x^2+1)$$