Math 114 Test 2 Solutions

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Problem 1.

(a) Suppose that if a car travels at v miles per hour then its fuel efficiency is $F(v) = 8 + 1.3v - .015v^2$ miles per gallon.

(i) What does the derivative $F'(v)$ represent, and what are its units?

Solution: The derivative $F'(v)$ is the rate at which fuel efficiency increases as your speed increases. The units are miles per gallon per mile per hour, which winds up working out to hours per gallon. (This is a little weird, but it actually makes sense: it's something like how many hours you save by burning an extra gallon of fuel).

(ii) Compute $F'(60)$. What does this tell you?

Solution: $F'(v) = 1.3 - .03v$ hours per gallon so $F'(60) = 1.3 - 1.8 = -.5$ hours per gallon. This tells us that if we are going sixty miles per hour, then increasing our speed by one mile per hour will reduce our gas milage by half a mile per gallon.

(b) Suppose that a factory produces widgets, and if p people work at the factory then they will produce a total of $W(p) = 30\sqrt{p}$ widgets.

(i) What does the derivative $W'(p)$ represent, and what are its units?

Solution: The derivative is the rate at which the number of widgets increases as we add more people to the factory (called the marginal product of labor). Its units are widgets per person.

(ii) Calculate $W'(9)$. What does this represent in the real world?

Solution: $W'(p) = \frac{15}{\sqrt{p}}$ so $W'(9) = 5$. So moving from nine people to ten people working at the factory will lead to the production of five extra widgets.

(c) A continuously-growing population of bacteria contains 1000 bacteria at 1:00 PM and 3000 bacteria at 3:00 PM. How many will it contain at 7:00 PM?

Solution: If $P(t)$ is the population of bacteria, we know that $P(t) = Ce^{rt}$. If we take 0 to be 1:00 PM, then we have $1000 = P(0) = Ce^{r0} = C$, so we know that $P(t) = 1000e^{rt}$.

Now we have $3000 = P(2) = 1000e^{2r}$, so we have $e^{2r} = 3$, and thus $2r = \ln(3)$ and $r = \ln(3)/2$.

Then $P(6) = 1000e^{6\ln(3)/2} = 1000e^{3\ln(3)}$. If you know about logarithms, you can further see that this is $1000 \cdot 3^3 = 27000$ bacteria at 7:00 PM.

Problem 2.

(a) When an object is flying through the air, it is slowed by air resistance. In many cases, the amount of deceleration is proportional to the square of the velocity. Write a differential equation to express this model, and identify the units of each variable and function you use.

Solution: The acceleration is the derivative of velocity, and in the opposite direction of the velocity. So we have $-v'(t) = kv(t)^2$.

(Technically the k could be a negative constant, but that's bad practice to leave things that way).

Here t has units of seconds, $v(t)$ has units meters per second, and k has units of $1/meter$.

(b) Check that the function $f(x) = e^{x^2}$ is a solution to the differential equation $y'' = 2xy' + 2x$.

Solution:

There was a typo in this question. I had intended to type $y'' = 2xy' + 2y$, in which case the answer would have been:

$$\begin{aligned} f'(x) &= 2xe^{x^2} \\ f''(x) &= 2e^{x^2} + 4x^2e^{x^2} \\ 2xf'(x) + 2f(x) &= 4x^2e^{x^2} + 2e^{x^2} = f''(x). \end{aligned}$$

However, with the question as written, the answer should be

$$\begin{aligned} f'(x) &= 2xe^{x^2} \\ f''(x) &= 2e^{x^2} + 4x^2e^{x^2} \\ 2xf'(x) + 2x) &= 4x^2e^{x^2} + 2x \neq f''(x). \end{aligned}$$

For the purposes of this test I'll mostly accept either answer.

(c) Compute $\frac{d}{dx}x^2e^{\tan(x)}$.

Solution:

$$\frac{d}{dx}x^2e^{\tan(x)} = 2xe^{\tan(x)} + x^2e^{\tan(x)}\sec^2(x).$$

Problem 3.

(a) Suppose we have the differential equation $f'(t) = f(t) - 2t$, and $f(1) = 3$. Use Euler's method with three steps to approximate $f(4)$.

Solution:

$$\begin{aligned} f(2) &\approx f(1) + f'(1)(2-1) = 3 + (1)(1) = 4 \\ f(3) &\approx f(2) + f'(2)(3-2) \approx 4 + (4-4)(3-2) = 4 \\ f(4) &\approx f(3) + f'(3)(4-3) \approx 4 + (4-6)(4-3) = 2. \end{aligned}$$

(b) Suppose we have the differential equation $f'(t) = t^2 + f(t)$, with $f(0) = 3$. Use Euler's method with four steps to approximate $f(4)$.

Solution: We have

$$\begin{aligned} f(1) &\approx f(0) + f'(0)(1-0) = 3 + 3(1) = 6 \\ f(2) &\approx f'(1)(2-1) + f(1) \approx 6 + (1+6)(1) = 13 \\ f(3) &\approx f'(2)(3-2) + f(2) \approx 13 + (4+13) = 30 \\ f(4) &\approx f'(3)(4-3) + f(3) \approx 30 + (9+30) = 69. \end{aligned}$$

Problem 4.

(a) Find a formula for y' in terms of x and y if $xy = x^2\sin(y)$.

Solution:

$$\begin{aligned} y + xy' &= 2x\sin(y) + x^2\cos(y)y' \\ y - 2x\sin(y) &= x^2\cos(y)y' - xy' \\ \frac{y - 2x\sin(y)}{x^2\cos(y) - x} &= y' \end{aligned}$$

(b) Find a tangent line to the curve given by $y^2 = x^3 + 5x + 3$ at the point $(1, -3)$.

Solution: We use implicit differentiation, and find that

$$2yy' = 3x^2 + 5$$

Thus at the point $(1, -3)$ we have

$$-6y' = 3 + 5$$
$$y' = \frac{-4}{3}$$

Thus the equation of our tangent line is

$$y + 3 = \frac{-4}{3}(x - 1).$$

(c) If $x^2y + xy = 3$ find a formula for $\frac{d^2y}{dx^2}$ in terms of x and y.

Solution: We get

$$\begin{aligned}
2xy + x^2y' + y + xy' &= 0 \\
x^2y' + xy' &= -2xy - y \\
y' &= \frac{-2xy - y}{x^2 + x} \\
y'' &= -\frac{(2y + 2xy' + y')(x^2 + x) - (2x + 1)(2xy + y)}{(x^2 + x)^2} \\
&= -\frac{\left(2y + 2x\frac{-2xy-y}{x^2+y} + \frac{-2xy-y}{x^2+x}\right)(x^2 + x) - (2x + 1)(2xy + y)}{(x^2 + x)^2}.
\end{aligned}$$

Problem 5.

(a) A hot air balloon lifts straight up off the ground above a point P. It is connected to a point on the ground 300 feet away from P by a rope, and the length of this rope is increasing by 10 feet per second. How fast is the balloon rising when it is 400 feet off the ground?

Solution: Let r be the length of the rope, and h the height of the balloon off the ground. Then we have $h = 400$ feet and $r' = 10$ feet per second. We also see that $h^2 + 300^2\text{ft}^2 = r^2$, which means that right now we have $r^2 = 300^2\text{ft}^2 + 400^2\text{ft}^2 = 500^2\text{ft}^2$ so $r = 500$ feet.

Now we compute that

$$\begin{aligned}
2hh' &= 2rr' \\
2 \cdot 400\text{ft} \cdot h' &= 2 \cdot 500\text{ft} \cdot 10\text{ft/s} \\
h' &= \frac{25}{2}\text{ft/s} = 12.5\text{ft/s}.
\end{aligned}$$

(b) A spot light is on the ground 36 ft away from a wall and a 5 ft tall person is walking towards the wall at a rate of 4 ft/sec. How fast is the height of the shadow changing when the person is 24 feet from the wall? Is the shadow increasing or decreasing in height at this time?

Solution: Let h be the height of the shadow, and d be the distance between the wall and the person. Then we want to find h'. We currently have $d = 24$. We know by similar triangles that $\frac{36-d}{36} = \frac{5}{h}$, which tells us that currently $h = 15$.

Then we have $d' = -4$. We compute

$$\begin{aligned} \frac{-d'}{36} &= \frac{-5h'}{h^2} \\ \frac{1}{9} &= \frac{-h'}{45} \\ h' &= \frac{-45}{9} = -5. \end{aligned}$$

Thus the shadow's height is decreasing by 5 feet per second.