Math 114 Test 3 Solutions

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Problem 1.

(a) Find $\lim_{x\to+\infty} x(\arctan(x) - \pi/4)$.

Solution: This problem was marred by an error. As written, we see that $\lim_{x\to+\infty} \arctan x = \pi/2$, so this is equal to $\lim_{x\to+\infty} x(\pi/2 - \pi/4) = +\infty$.

The problem was supposed to ask for $\lim_{x\to+\infty} x(\arctan(x) - \pi/2)$. This is a "$0\cdot\infty$" indeterminate form. So we rewrite it as

$$\lim_{x\to+\infty} \frac{\arctan(x) - \pi/2}{1/x}.$$

The top and the bottom both go to zero, so we can use L'Hospital's Rule, and get

$$\begin{aligned}\lim_{x\to+\infty} \frac{\arctan(x) - \pi/2}{1/x}. - &\lim_{x\to+\infty} \frac{1/(1+x^2)}{-1/x^2} \\ = &\lim_{x\to+\infty} \frac{-x^2}{1+x^2} \\ = &\lim_{x\to+\infty} \frac{-1}{1/x^2+1} = -1.\end{aligned}$$

(b) Find $\lim_{x\to 2} \frac{x^3 - 3x - 1}{2x^3 - 3x^2 + 3}$.

Solution:

$$\lim_{x\to 2} \frac{x^3 - 3x - 1}{2x^3 - 3x^2 + 3} = \frac{1}{7}.$$

(c) Find $\lim_{x\to 0} \frac{\sin(x)}{e^{3x} - 1}$.

Solution: The top and bottom both approach zero, so we can use L'Hospital's Rule.

$$\lim_{x\to 0} \frac{\sin(x)}{e^{3x} - 1} = \lim_{x\to 0} \frac{\cos(x)}{3e^{3x}} = \frac{1}{3}.$$

Problem 2.

(a) Compute $f'(x)$ where $f(x) = \arcsin \log_3(x^2)$.

Solution:

$$f'(x) = \frac{1}{\sqrt{1 - \log_3(x^2)^2}} \frac{1}{x^2 \ln(3)} 2x$$

(b) Compute $g'(\pi/6)$ where $g(x) = 9^{\sin(x)}$.

Solution:

$$g'(x) = 9^{\sin(x)} \ln(9) \cos(x)$$

so

$$g'(\pi/6) = 9^{\sin(\pi/6)} \cdot 2\ln(3)\cos(\pi/6) = 6\ln(3) \cdot \frac{\sqrt{3}}{2} = 3\ln(3)\sqrt{3} \approx 5.71.$$

(c) Find the tangent line to $h(x) = e^{x^2-1}$ at -1.

Solution: We have

$$h'(x) = e^{x^2-1}(2x)$$

so

$$h'(-1) = e^0(-2) = -2.$$

Further $h(-1) = e^0 = 1$.

Thus the equation of the tangent line is

$$y - 1 = -2(x+1)$$

or

$$y = -2x - 1.$$

Problem 3.

(a) Find the derivative of $f(x) = x^{\ln(x)}$

Solution:

We have

$$\begin{aligned} \ln y &= \ln(x)^2 \\ y'/y &= 2\ln(x)\frac{1}{x} \\ y' &= x^{\ln(x)}\frac{2\ln(x)}{x}. \end{aligned}$$

(b) Let $g(x) = x^7 + x^3 + x - 3$. Find $(g^{-1})'(0)$.

Solution: Plugging in numbers, we see that $g(1) = 1 + 1 + 1 - 3 = 0$, so $g^{-1}(0) = 1$. Then by the Inverse Function Theorem we have $(g^{-1})'(0) = \frac{1}{g'(1)}$. But

$$\begin{aligned} g'(x) &= 7x^6 + 3x^2 + 1 \\ g'(1) &= 7 + 3 + 1 = 11 \end{aligned}$$

Thus by the inverse function theorem we have

$$(g^{-1})'(0) = \frac{1}{11}.$$

(c) Compute the following. In all cases your answers should be exact, with no decimals, and no logs or exponentials or trig functions..

$$\log_2(8) + \log_2(6) - \log_2(3)$$

Solution: $3 + \log_2(2) = 4$

$$\arctan(\sqrt{3}) =$$

Solution: $\pi/3$

$$\sin(\arctan(4/3)) =$$

Solution: $4/5$

Problem 4.

(a) Show that the function $f(x) = e^x + x$ has a real root. That is, show there is a real number c such that $f(c) = 0$.

Solution: Since f is made of reasonable pieces it must be continuous everywehre.

$$f(0) = e^0 + 0 = 1 > 0$$
$$f(-1) = e^{-1} - 1 = \frac{1}{e} - 1 < 0.$$

We have $f(-1) < 0 < 1$, and f is continuous on $[-1, 0]$, so by the Intermediate Value Theorem there is some c between -1 and 0 with $f(a) = 0$.

(b) Find the general form of an antiderivative for $e^x + 2x$.

Solution: $e^x + x^2 + C$.

(c) Find y' if $y^x = x$.

Solution:

$$\begin{aligned} x \ln y &= \ln x \\ \ln(y) + \frac{xy'}{y} &= \frac{1}{x} \\ \frac{xy'}{y} &= \frac{1}{x} - \ln(y) \\ y' &= \frac{y}{x^2} - \frac{y\ln(y)}{x}. \end{aligned}$$

Problem 5.

(a) Use two iterations of Newton's Method starting at 1 to estimate a root of $f(x) = x^3 + x - 3$.

Solution:

We start with $x_0 = 1$. We need to find a root of $f(x) = x^3 + x - 3$, so we also need $f'(x) = 3x^2 + 1$.

$$\begin{aligned} x_1 &= 1 - \frac{-1}{4} = \frac{5}{4} \\ x_2 &= \frac{5}{4} - \frac{(5/4)^3 + 5/4 - 3}{3(5/4)^2 + 1} = \frac{5}{4} - \frac{125/64 + 80/64 - 192/64}{75/16 + 16/16} = \frac{5}{4} - \frac{13/64}{91/16} \\ &= \frac{5}{4} - \frac{13}{364} = \frac{442}{364} = \frac{17}{14} \approx 1.21 \end{aligned}$$

(b) Find the formula for the quadratic approximation of $g(x) = x^5 + x$ near 2.

Solution: We have

$$\begin{aligned} g(x) &= x^5 + x & g(2) &= 34 \\ g'(x) &= 5x^4 + 1 & g'(2) &= 81 \\ g''(x) &= 20x^3 & g''(1) &= 160 \end{aligned}$$

and thus we have

$$g(x) \approx 34 + 81(x-2) + 80(x-2)^2.$$